Universal Analytic Gröbner Bases and Tropical Geometry

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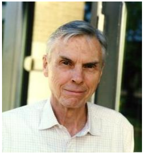
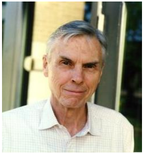
Algebraic geometry ⟵ GAGA ⟶ Analytic geometry



Rigid geometry ⟷ Tate ⟷ Algebraic geometry ⟷ GAGA ⟷ Analytic geometry

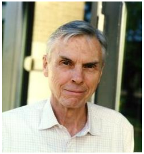
Tropical analytic geometry

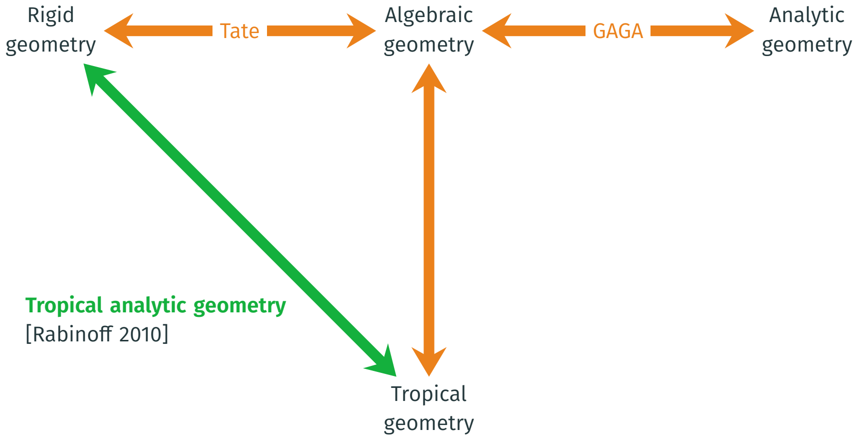


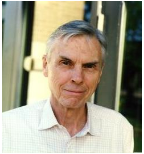
Rigid geometry ← Tate → Algebraic geometry ← GAGA → Analytic geometry

Algebraic geometry ↕ Tropical geometry

Tropical analytic geometry







Rigid geometry ⟵ Tate ⟶ Algebraic geometry ⟵ GAGA ⟶ Analytic geometry

This work:

- Effective theory of universal analytic Gröbner bases
- Application to tropical geometry
- Towards effective tropical analytic geom.

Tropical analytic geometry
[Rabinoff 2010]

Tropical geometry

Tropical varieties

K field with valuation (e.g. $\mathbb{Q}((t)), \mathbb{Q}_p \dots$)

$I \subseteq K[\boldsymbol{X}]$ ideal

Definition with valuation:
$\mathrm{trop}(I) = \mathrm{clo}\{\mathrm{val}(\boldsymbol{x}) : \boldsymbol{x} \in V(I)\}$

Newton-Puiseux, Hensel, etc.

Fund. th. of tropical geometry

Definition with initial forms:

System of weights: $\boldsymbol{w} = (w_0, \dots, w_n) \in \mathbb{R}^{n+1}$

$\boldsymbol{w}$-valuation: $\mathrm{val}_{\boldsymbol{w}}(aX_1^{\alpha_1} \cdots X_n^{\alpha_n}) = w_0 \mathrm{val}(a) + w_1\alpha_1 + \cdots + w_n\alpha_n$

$\mathrm{init}_{\boldsymbol{w}}(f)$ = sum of terms with minimal $\boldsymbol{w}$-valuation

$\mathrm{init}_{\boldsymbol{w}}(I) = \langle \mathrm{init}_{\boldsymbol{w}}(f) : f \in I \rangle$

$\mathrm{trop}(I) = \{\boldsymbol{w}$ such that $\mathrm{init}_{\boldsymbol{w}}(I)$ does not contain a monomial$\}$

$x = \alpha t^1 + \cdots, y = \beta t^1 + \cdots$

$f(x,y) = \alpha^2 t^2 + \beta^2 t^4 - t^4 + \cdots$

$= \alpha^2 t^2 + \cdots \neq 0$

$\mathrm{init}_{(1,1)}(I) = \langle x^2 \rangle$

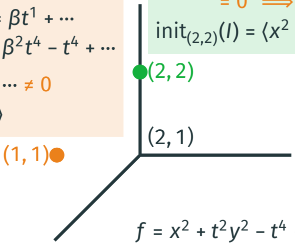
$x = \alpha t^2 + \cdots, y = \beta t^2 + \cdots$

$f(x,y) = \alpha^2 t^4 + \beta^2 t^6 - t^4 + \cdots$

$= (\alpha^2 - 1)t^4 + \cdots$

$= 0 \implies \alpha^2 - 1 = 0$

$\mathrm{init}_{(2,2)}(I) = \langle x^2 - t^4 \rangle$



$f = x^2 + t^2y^2 - t^4 \in \mathbb{Q}((t))[x,y]$

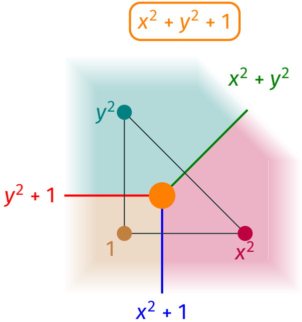
$I = \langle f \rangle$

Gröbner fan:

- Partition of $\mathbb{R}^n$ according to

$$\boldsymbol{a} \sim \boldsymbol{b} \iff \mathrm{init}_{\boldsymbol{a}}(I) = \mathrm{init}_{\boldsymbol{b}}(I)$$

- Finite union of rational cones
- Maximal dim. = term orders
- Lower dim. = boundaries = collisions
- Contains the tropical variety

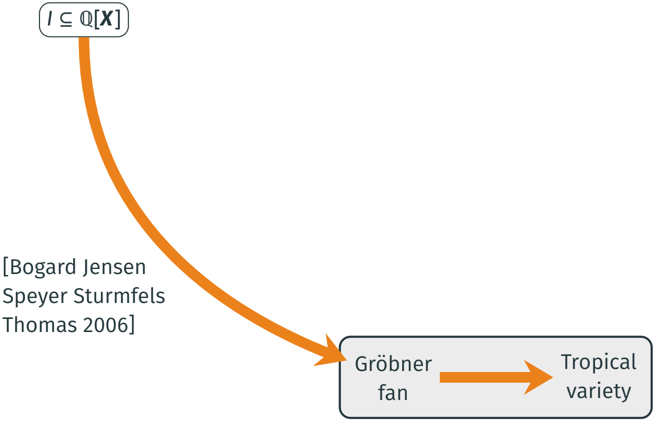


Tropical variety of I (without valuations)

1. Compute a universal GB of I
2. Compute its Newton polytope
3. Compute the Gröbner fan of I
4. For each non-maximal cone $\mathcal{C}$, pick a $\boldsymbol{w} \in \mathcal{C}$ and test if $\mathrm{init}_{\boldsymbol{w}}(I)$ contains a monomial

In practice: traverse only the necessary parts of the Gröbner fan

[Bogard Jensen Speyer Sturmfels Thomas 2006]


$I \subseteq \mathbb{Q}[\boldsymbol{X}]$
[Bogard Jensen Speyer Sturmfels Thomas 2006]
Gröbner fan
Tropical variety

Computing tropical varieties with valuation

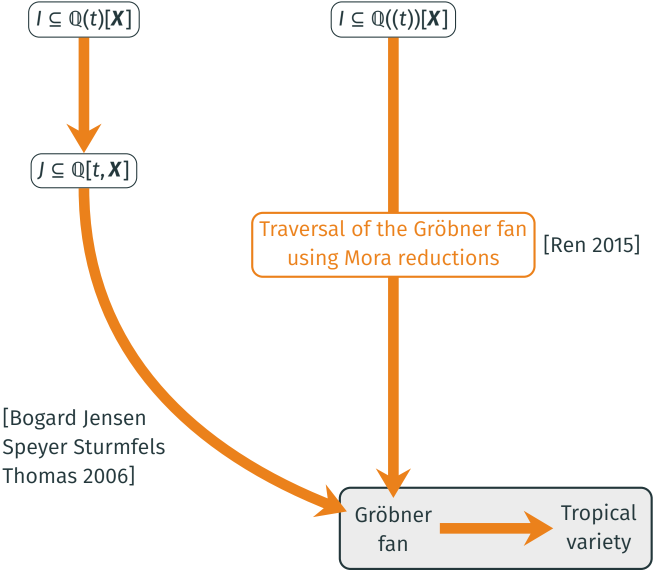
$I \subseteq \mathbb{Q}(t)[\boldsymbol{X}]$

$J \subseteq \mathbb{Q}[t, \boldsymbol{X}]$

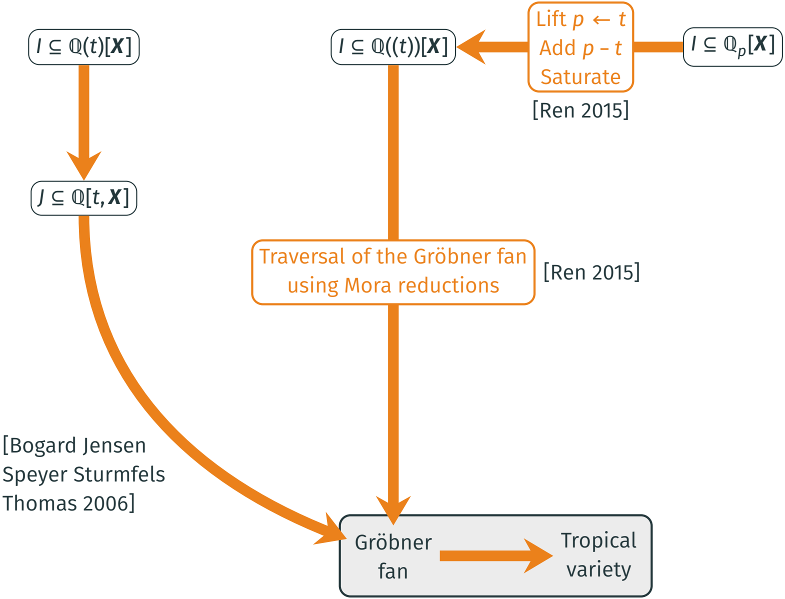
[Bogard Jensen Speyer Sturmfels Thomas 2006]

Gröbner fan $\longrightarrow$ Tropical variety

Computing tropical varieties with valuation



Computing tropical varieties with valuation



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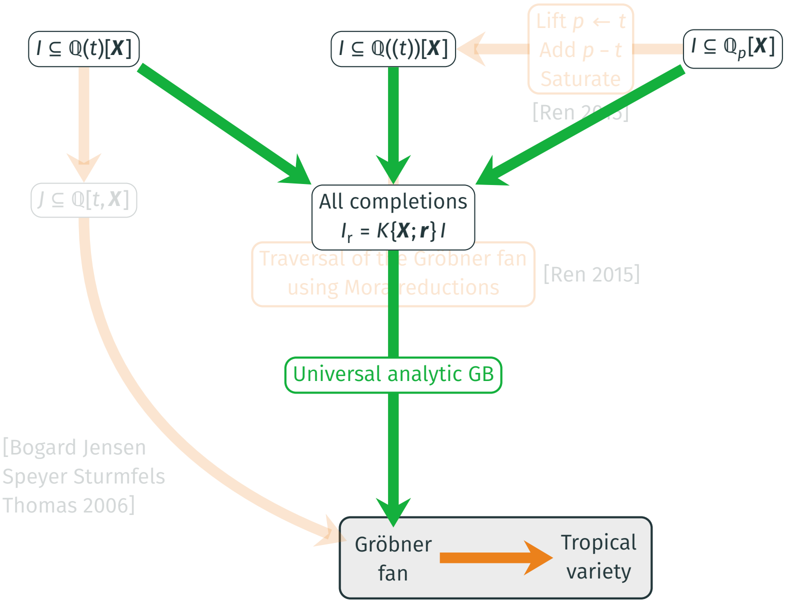


Tate series

Definition: convergent series with coefficients in a valued field or ring ($\mathbb{Q}(t), \mathbb{Q}((t)), \mathbb{Q}_p$ ...)

$$K\{\boldsymbol{X};\boldsymbol{r}\} = \left\{ \sum_{\alpha\in\mathbb{N}^n} a_\alpha \boldsymbol{X}^\alpha \text{ with } \mathrm{val}(a_\alpha) - \alpha\cdot\boldsymbol{r} \xrightarrow{|\alpha|\to\infty} \infty \right\}$$

$\boldsymbol{r} = (r_1, \dots, r_n)$: convergence radii

- If $\boldsymbol{r} = (0, \dots, 0)$, equivalent: $a_\alpha \to 0$
- If $\boldsymbol{r} \in \mathbb{Z}^n$, equivalent to change of variable $K\{\boldsymbol{X};\boldsymbol{r}\} = K\{(X_i/p^{r_i});(0)\}$
- $K[\boldsymbol{X}] = K\{\boldsymbol{X};\infty\}$ (everywhere convergent)

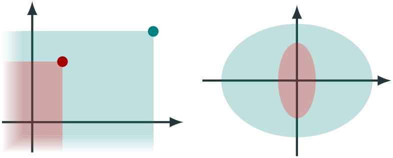
Term ordering:

$$a\boldsymbol{X}^\alpha <_{\boldsymbol{r}} b\boldsymbol{X}^\beta \iff \begin{cases} \mathrm{val}(a) - \boldsymbol{r}\cdot\alpha > \mathrm{val}(b) - \boldsymbol{r}\cdot\beta \\ \text{or they are equal and } \boldsymbol{X}^\alpha < \boldsymbol{X}^\beta \end{cases}$$

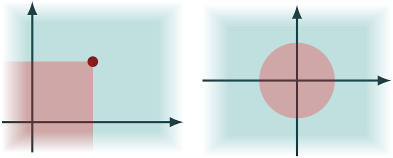
- Every Tate series has a leading term
- Every Tate series ideal has a finite Gröbner basis
- Different $\boldsymbol{r}$ give different leading terms and Gröbner bases

Overconvergence

Fact: If $\boldsymbol{r} \leq \boldsymbol{s}$, then $K\{\boldsymbol{X};\boldsymbol{s}\} \subseteq K\{\boldsymbol{X};\boldsymbol{r}\}$:



$K\{\boldsymbol{X};(4,3)\} \subseteq K\{\boldsymbol{X};(1,2)\}$



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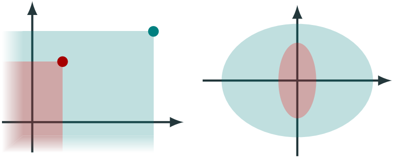

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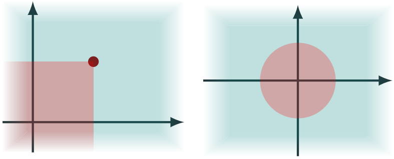

$K\{\boldsymbol{X}; (\infty,\infty)\} = K[\boldsymbol{X}] \subseteq K\{\boldsymbol{X}; (1,2)\}$

Theorem (Caruso, Vaccon, V. 2022)
Let $\boldsymbol{r} \ge \boldsymbol{s}$, $I \subset K\{\boldsymbol{X}; \boldsymbol{r}\}$ and $I_{\boldsymbol{s}} = K\{\boldsymbol{X}; \boldsymbol{s}\} I$ (completion of the ideal).
Then $I_{\boldsymbol{s}}$ admits a Gröbner basis comprised only of elements of $K\{\boldsymbol{X}; \boldsymbol{r}\}$.
In particular, the completion of a polynomial ideal has a polynomial basis.

Key component: Mora's reduction algorithm

Input: $G \subset K\{\boldsymbol{X}; \boldsymbol{r}\}$, $f \in K\{\boldsymbol{X}; \boldsymbol{r}\}$

Output: $h, u \in K\{\boldsymbol{X}; \boldsymbol{r}\}$, such that:

- uf reduces to h and is then irreducible modulo G (in $K\{\boldsymbol{X}; \boldsymbol{s}\}$)
- $\mathrm{LT}_{\boldsymbol{s}}(u) = 1$, or equivalently, u is invertible in $K\{\boldsymbol{X}; \boldsymbol{s}\}$

$$\overbrace{a_0 \boldsymbol{X}^{\alpha_0}}^{\mathrm{LT}_{\boldsymbol{r},\leq}(f)}$$

$$f = \underbrace{a_0 \boldsymbol{X}^{\alpha_0} + a_1 \boldsymbol{X}^{\alpha_1} + \cdots + a_k \boldsymbol{X}^{\alpha_k}}_{\text{minimal } \mathrm{val}(a_i) - \alpha_i \cdot \boldsymbol{r}} + a_{k+1} \boldsymbol{X}^{\alpha_{k+1}} + \cdots \quad \text{with } \mathrm{val}(a_i) - \alpha_i \cdot \boldsymbol{r} \xrightarrow{|\alpha_i| \to \infty} \infty$$

$$f = \underbrace{\overbrace{a_0\boldsymbol{X}^{\alpha_0}}^{\mathrm{LT}_{\boldsymbol{r},\leq}(f)} + a_1\boldsymbol{X}^{\alpha_1} + \cdots + a_k\boldsymbol{X}^{\alpha_k}}_{\text{minimal } \mathrm{val}(a_i) - \alpha_i \cdot \boldsymbol{r} \;=\; \mathrm{init}_{(-1,\boldsymbol{r})}(f)} + a_{k+1}\boldsymbol{X}^{\alpha_{k+1}} + \cdots \quad \text{with } \mathrm{val}(a_i) - \alpha_i \cdot \boldsymbol{r} \xrightarrow{|\alpha_i| \to \infty} \infty$$

Theorem (Vaccon, V. 2023)
Let $I \subseteq K[\boldsymbol{X}]$, and $\boldsymbol{w} \in \mathbb{Q}^{n+1}$ a system of weights.
Let $\boldsymbol{r} = (-w_1/w_0, \ldots, -w_n/w_0)$ and $I_{\boldsymbol{r}} = I\,K\{\boldsymbol{X};\boldsymbol{r}\}$ the corresponding completion, then

$$\mathrm{init}_{\boldsymbol{w}}(I) = \mathrm{init}_{\boldsymbol{w}}(I_{\boldsymbol{r}}) \cap K[\boldsymbol{X}].$$

This is a local result, which translates globally as:

$$V_{\mathrm{trop}}(I) = \bigsqcup_{\boldsymbol{s} \in \mathbb{Q}^n} \mathrm{trop}(I_{\boldsymbol{s}}).$$

Theorem (Vaccon, V. 2023)
Let G be a Gröbner basis of $I_{\boldsymbol{r}}$, then

$$\mathrm{init}_{\boldsymbol{w}}(I_{\boldsymbol{r}}) = \langle \mathrm{init}_{\boldsymbol{w}}(g) : g \in G \rangle.$$

Theorem (Caruso, Vaccon, V. 2022; Vaccon, V. 2023)
Let $I \subseteq K[\boldsymbol{X}]$ be a homogeneous ideal.
There exists a finite subset $G \subseteq I$ s.t. for all $\boldsymbol{r} \in \mathbb{Q}^n$, G is a Gröbner basis of $I_{\boldsymbol{r}} = I\,K\{\boldsymbol{X};\boldsymbol{r}\}$.
Furthermore:

- this is independent of the order used for breaking ties
- G can be computed

As a consequence, a homogeneous polynomial ideal always has a finite Gröbner fan.

Why homogeneous?

- Key question: how does $\mathrm{init}_{\boldsymbol{a}}(I) = \mathrm{init}_{\boldsymbol{b}}(I)$ relate to $\mathrm{init}_{\boldsymbol{a}}(G) = \mathrm{init}_{\boldsymbol{b}}(G)$?
- Usually, this is answered by taking reduced Gröbner bases
- We can have reduced GB (with the usual algorithm),
  or overconvergent GB (using Mora's algorithm)... but not both in general
- For homogeneous ideals, reduced overconvergent bases exist

Tropical variety with valuation

I. $I \subseteq K[X]$ homogeneous ideal.

O. the tropical variety of I, given as a union of rational cones

1. compute a universal analytic Gröbner basis G of I
2. get all the maximal dimensional cones in the Gröbner fan
3. compute the rest of the cones
4. for each non-maximal cone $\mathcal{C}$
   - 4.1 pick a $\boldsymbol{w} = (-1, \boldsymbol{r}) \in C$
   - 4.2 Then $\mathrm{init}_{\boldsymbol{w}}(G)$ is a Gröbner basis of $\mathrm{init}_{\boldsymbol{w}}(I_{\boldsymbol{r}})$
   - 4.3 test if $\mathrm{init}_{\boldsymbol{w}}(I)$ contains a monomial
   - 4.4 conclude whether $\boldsymbol{w} \in \mathrm{trop}(I)$ and therefore $\mathcal{C} \subseteq \mathrm{trop}(I)$.

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Mora's reduction
UAGB algo.

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UAGB algo.

Newton polytope

Discrete geometry

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Mora's reduction
UAGB algo.
Newton polytope
Discrete geometry
Th. on Tate GB
Saturation

Tropical variety with valuation

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Mora's reduction
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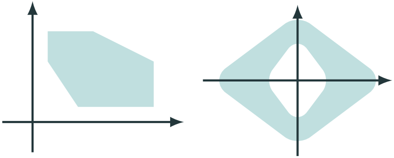
Saturation

- Proof of concept showing that the main algorithmic ingredients are in place: universal Gröbner basis, Gröbner fan, connection to the tropical variety
- Next task: transposing the advanced traversal techniques used in the classical setting

So far, we have shown that for (poly)disks:

- We can compute overconvergent Gröbner bases
- We can compute a universal analytic Gröbner basis

What about more general convergence conditions?



Questions:

- Local: can we compute overconvergent Gröbner bases?
- Global: Can we compute a universal analytic Gröbner basis?

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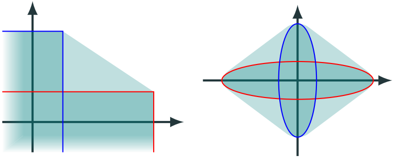
Example: upper polyhedral domains:



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Th. (Vaccon V. 2023) **YES**

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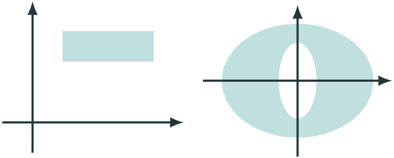
Example: annuli (with Laurent terms):



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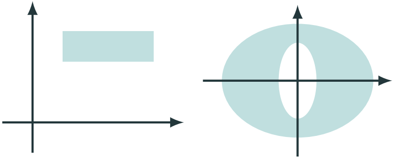

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Thank you for your attention!