Using Signature Gröbner Bases to Find Short Ideal Representations

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Example:

- Definition: an inner inverse of a matrix A is a matrix B such that $ABA = A$
- Theorem: if A is invertible with inverse C and if B is an inner inverse of A, then $B = C$
- Proof: easy exercise ■

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How to prove this with Gröbner bases?

- Objects → generators of a free algebra: $K\langle a, b, c\rangle$
- Axioms → ideal: $I = \langle ac - 1, ca - 1\rangle + \langle aba - a\rangle$
- Theorem → Ideal Membership Problem: $b - c \in I$?
- New proof: $b - c$ reduces to zero modulo a Gröbner basis of I
- Proof: trust me ■

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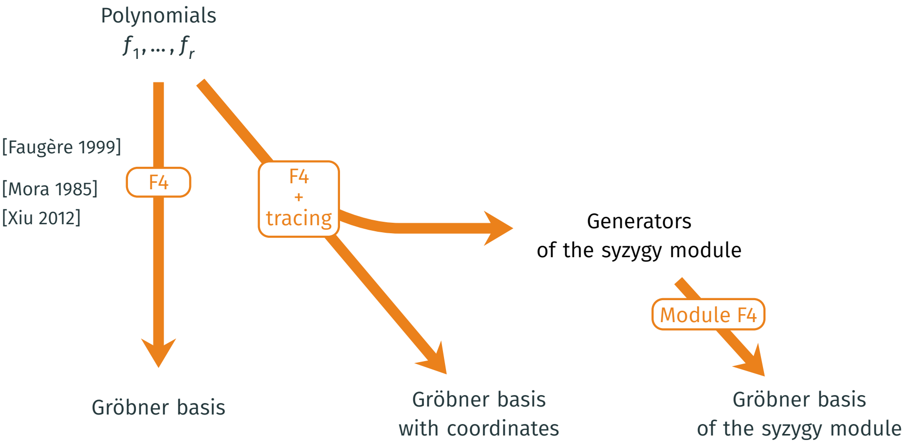
- Proof: cofactor representation:

$$c - b = b\,(ac - 1) - c\,(ac - 1) - c\,(aba - a)\,c + (ca - 1)\,bac. \qquad \blacksquare$$



Ideal Membership Problem

"Does there exist (a_i, b_j) such that $p = a_1 f_{i_1} b_1 + \dots + a_N f_{i_N} b_N$?"

IMP with certificate

"Compute (a_i, b_j) such that $p = a_1 f_{i_1} b_1 + \dots + a_N f_{i_N} b_N$"

Module of syzygies

"Find all (a_i, b_j) such that $a_1 f_{i_1} b_1 + \dots + a_N f_{i_N} b_N = 0$"

Gröbner bases and signature Gröbner bases

Polynomials
$f_1, \dots, f_r$

F5/GVW

[Faugère 2002] [Gao Volny Wang 2010]
[Eder Faugère 2017] [Lairez 2022]

[Hofstadler V. 2022, 2023]

Gröbner basis
with signatures

Gröbner basis

Gröbner basis
with coordinates

Gröbner basis
of the syzygy module

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Back to the example

Question: does $b - c$ lie in the ideal $\langle ac - 1, ca - 1, aba - a\rangle$ in $K\langle a, b, c\rangle$?

	Gröbner basis element	Cofactors
$\boxed{1}$	$ac - 1$	$\boldsymbol{e}_1$
$\boxed{2}$	$ca - 1$	$\boldsymbol{e}_2$
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4. $ab - 1$ with signature $\mathbf{e}_3 c \longrightarrow f_3 c \qquad - (ab - 1) = abac - ac - ab + 1 \quad \neq 0$

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\begin{aligned}
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&f_3 c - abf_1 &&- (ab - 1) = -ac + 1 && \neq 0\\
&f_3 c - abf_1 + f_1 &&- (ab - 1) = 0 &&
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$$f_3 c - (ab-1) = abac - ac - ab + 1 \neq 0$$
$$f_3 c - abf_1 - (ab-1) = -ac + 1 \neq 0$$
$$f_3 c - abf_1 + f_1 - (ab-1) = 0$$
5. $-c+b$ with signature $c\boldsymbol{e}_3 c \longrightarrow$
$$c(f_3 c - abf_1 + f_1) - (-c+b) = cab - b \neq 0$$
$$cf_3 c - cabf_1 + cf_1 - f_2 b - (-c+b) = 0$$

Short cofactor representations

Cofactor representations are not unique!

Example: $b - c = c(aba - a)c + (ca - 1)bac + b(ac - 1) - c(ac - 1)$

$$= c(aba - a)c - (ca - 1)b - cab(ac - 1) + c(ac - 1)$$

Short proofs are important for proof analysis.

Example: $b - c \in I = \langle ac - 1, ca - 1, aba - a, bab - b, ab - ed, ba - de \rangle \subseteq K\langle a, b, c, d, e \rangle$

($b : A^\dagger$ (Moore-Penrose inverse), $c : A^{-1}, d : A^*, e : (A^\dagger)^*$)

$$\begin{aligned} b - c = {} & b(ac - 1) - c(ac - 1) + (ca - 1)bac + c(aba - a)c \\ & - be(bab - b) + (bab - b)eb - b(ab - ed)eb + be(ba - de)a \end{aligned}$$

This work

- Algorithm for finding short (and often shortest) representations
- Also works in the commutative case
- Reduces to and generalizes a classical linear algebra problem
- Main tools: signatures and linear optimization

On the edge of decidability

Fact: the IMP over non-commutative polynomials is not decidable

Theorem (Hofstadler, V. 2023)
The problem of, given $f_1, \dots, f_r, f \in K\langle X\rangle$ and $N \in \mathbb{N}$, deciding whether f has a cofactor representation of length at most N in the f_i,
i.e., whether there exists $a_k, b_k \in \langle X\rangle$, $i_k \in \{1, \dots, r\}$ such that

$$f = a_1 f_{i_1} b_1 + \cdots + a_N f_{i_N} b_N,$$

is decidable.

Difficulty: no bound on the degrees of a_k and b_k

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For minimal representations, we can say more!

Rewriting
Example: given $f = x + xy$, $g = y + z$:

- $f - xg = x + xy - xy - xz$ is a rewriting of f by g
- $f + xg = x + 2xy + xz$ is also a rewriting of f by g
- $f + yg = x + xy + y^2 + yz$ is not a rewriting

- Any minimal representation of f by $f_1, \dots, f_r$ of length N can be expressed as N successive rewritings starting with f and ending with 0
- There is a bound on the degree of terms appearing in a rewriting!

Algorithm

I: $f, f_1, \dots, f_r \in K\langle X\rangle$, $N \in \mathbb{N}$

O: a (minimal) cofactor representation of f of length $\leq N$ if one exists

1. Compute the degree bound $D = \deg(f) + N \max \deg(f_i)$
2. Compute the set L of all polynomials af_ib with $i \in \{1, \dots, n\}$, $a, b \in \langle X\rangle$, $\deg(af_ib) \leq D$
3. Look for a K-linear combination of elements of L equal to f with $\leq N$ nonzero summands

Observations:

- The bulk of the computation is in the last step
- That step is difficult but decidable (if only by bruteforce)
- The input for that step is huge but finite
- The algorithm is not practical for anything but trivial examples

So we need:

- A better algorithm for solving the last step: linear programming
- A better bound on the search space: signatures

Min-RVLS (Minimum Relevant Variables in Linear Systems):

I: $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m, N \in \mathbb{N}$

O: $y \in \mathbb{Q}^n$ with $Ay = b$ and at most N nonzero coordinates if one exists

This is a difficult problem: NP-complete, hard to approximate.

Basis pursuit: ℓ_1 relaxation + Linear Programming [Chen, Donoho, Saunders 2001]

I. $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m, N \in \mathbb{N}$

O. $y \in \mathbb{Q}^n$ minimizing $\sum_{i=1}^n |y_i|$ under the constraint $Ay = b$.

Algorithm

I: $f, f_1, \dots, f_r \in K\langle X\rangle$, σ a signature

O: a cofactor representation of f with signature $\leq \sigma$ and minimal 1-norm (if it exists)

1. Compute the set L of all polynomials $a_k f_{i_k} b_k$ with $a_k \boldsymbol{e}_{i_k} b_k \leq \sigma$
2. Put them in a matrix A
3. Solve the linear problem:
   find $\begin{pmatrix} p \\ m \end{pmatrix}$ minimizing $\sum_i p_i + \sum_i m_i$ under the constraints $\begin{pmatrix} A & -A \end{pmatrix} \begin{pmatrix} p \\ m \end{pmatrix} = f$ and $p, m \geq 0$
4. Return $\sum_k (p_k - m_k) a_k \boldsymbol{e}_{i_k} b_k$

Observations:

- Deciding whether the search space is empty requires finding *any* solution of the linear system
- Doing so is essentially equivalent to computing a signature Gröbner basis up to signature σ (Matrix-F5 algorithm)

Symbolic preprocessing

Observation: if α and β have disjoint support, then

$$\|\alpha + \beta\| = \|\alpha\| + \|\beta\| \qquad \text{(both for 0-“norm” and 1-norm)}$$

Theorem (Hofstadler, V. 2023)

Let:

- H be a Gröbner basis of the module of syzygies up to some signature σ
- α be a cofactor representation of f in terms of $f_1, \ldots, f_r$ with $\mathrm{sig}(\alpha) \leq \sigma$
- β be a representation of f, shortest or 1-norm minimal among those with sig $\leq \sigma$

Then α can be rewritten to β by H, and the signature of every rewriter is $\leq \sigma$.

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Algorithm: Symbolic preprocessing

I: σ, H, α as above

O: a generating set B of $V \subset \mathrm{Syz}(f_1, \dots, f_r)$ such that for any optimal β, $\beta \in \alpha + V$.

1. $B \leftarrow \emptyset$, TODO $\leftarrow$ Support of α
2. While TODO is not empty
   - 2.1 Pick a monomial $\mu \in$ TODO
   - 2.2 Take all the multiples $a\gamma b$ of elements in H with signature $\leq \sigma$ and with μ in their support
   - 2.3 Add them to V and their support to TODO

Algorithm

I: $f, f_1, \dots, f_r \in K\langle X\rangle$, σ a signature, α a representation with signature $\leq \sigma$

O: a cofactor representation of f with signature $\leq \sigma$ and minimal 1-norm (it exists!)

1. Compute a Gröbner basis of $\mathrm{Syz}(f_1, \dots, f_r)$ up to signature σ
2. Compute the search space $B = \left\{a_1 \boldsymbol{e}_{i_1} b_1,\ a_2 \boldsymbol{e}_{i_2} b_2, \dots\right\}$ (symbolic preprocessing)
3. Prune B using further criteria and heuristics
4. Put B and α in a matrix A
5. Solve the linear problem:
   find $\begin{pmatrix} p \\ m \end{pmatrix}$ minimizing $\sum_i p_i + \sum_i m_i$ under the constraints $\begin{pmatrix} A & -A \end{pmatrix}\begin{pmatrix} p \\ m \end{pmatrix} = f$ and $p, m \geq 0$
6. Return $\sum_k (p_k - m_k) a_k \boldsymbol{e}_{i_k} b_k$

About the ℓ_1-relaxation

- With the 1-norm, $x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3$ is smaller than $35z\boldsymbol{e}_4 - \boldsymbol{e}_5$.
- Experimentally, the algorithm can still find shorter proofs than the best computer proofs
- In practice, in many examples the coefficients are small, bringing the two sparsity measures closer together
- In particular, a lot of examples are totally unimodular (e.g. pure difference binomials)
- Then the coefficients of minimal representations are 0 or ±1
- In that case the algorithm is guaranteed to return a sparsest solution

- In the other direction, the linear programming approach opens the way to thinner metrics
- For example, it is possible to weigh the 1-norm with the degree of the terms

Experimental data

Example	#gens	deg	GB	SigGB	LP	Mtx size	Pruning ratio
SVD	32	3	51	39	25	118 k × 328 k	0.83
ROL	28	5	80	39	30	22 k × 56 k	0.55
ROL-2	28	5	20	21	15	23 k × 60 k	0.56
ROL-3	28	5	49	44	31	18 k × 46 k	0.53
ROL-4	28	5	59	46	33	64 k × 137 k	0.58
ROL-5	28	5	28	30	22	31 k × 80 k	0.60
ROL-6	28	5	39	39	30	21 k × 55 k	0.56
ROL-7	40	9	85	23	17	17 k × 46 k	0.54
ROL-8	44	7	241	19	17	249 k × 560 k	0.58
Hartwig-4	23	15	316	54	46	349 k × 1 460 k	0.84
Hartwig-5	26	15	99	43	35	398 k × 1 374 k	0.84
Hartwig-6	24	15	86	33	29	217 k × 808 k	0.84
Ker	12	3	49	34	23	51 k × 129 k	0.90
SMW	36	7	63	42	39	44 k × 94 k	0.83
Sum	20	3	313	178	85	11 k × 17 k	0.93

Conclusion

What we presented

- New approach for computing short proofs of ideal membership
- Generalization of Min-RVLS to algebraic systems
- Combination of signature techniques and linear programming
- Flexible for other metrics
- Also works in the commutative case

Future work

- Choice of a signature ordering
- More efficient representations using additional generators ("lemmas")

More details and references

- Hofstadler and Verron, *Signature Gröbner bases, bases of syzygies and cofactor reconstruction in the free algebra* (JSC 2022) `ArXiV:2107.14675`
- Hofstadler and Verron, *Signature Gröbner bases in free algebras over rings* (ISSAC 2023) `ArXiV:2302.06483`
- Hofstadler and Verron, *Short proofs of ideal membership*, `ArXiv:2302.02832`

Thank you for your attention!