Computing tropical varieties using Tate series

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Tropical varieties: an example

Consider the polynomial equation:

$$x^2 + t\,y^2 + t^2 = 0$$

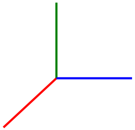
over $\mathbb{Q}(t)[x, y]$.

How to find series solution? Newton-Puiseux algorithm:

1. Make an ansatz: $x = t^a + \cdots$, $y = t^b + \cdots$
2. Plug in: $t^{2a} + \cdots + t^{1+2b} + \cdots + t^2 = 0$
3. Force a collision to find a and b:

$$\begin{cases} & 2a = 1 + 2b \leq 2 \\ \text{or} & 2a = 2 \leq 1 + 2b \\ \text{or} & 1 + 2b = 2 \leq 2a \end{cases}$$

4. Solve:



Tropical varieties: definitions

K: field with valuation (e.g. $\mathbb{Q}((t))$ or $\mathbb{Q}_p$), $I \subset K[\boldsymbol{X}]$

Definition using the image of the valuation:

$$V_{\text{trop}}(I) = \{\text{val}(\boldsymbol{s}) : s \in V_{\bar{K}}(I)\}$$

Definition using the tropical semi-ring: for $f = \sum a_{\alpha \in S} a_\alpha \prod x_i^{\alpha_i}$, define

$$f_{\text{trop}} = \min\left(\text{val}(a_\alpha) \sum_{\alpha_i} x_i : \alpha \in S\right)$$

and

$$V_{\text{trop}}(f) = \{\boldsymbol{w} \in \mathbb{Q}^n : f_{\text{trop}} \text{ is not differentiable at } \boldsymbol{w}\}$$

Then:

$$V_{\text{trop}}(I) = \bigcap_{f \in I} V_{\text{trop}}(f)$$

Definition using initial ideals: for $\boldsymbol{w} \in \mathbb{Q}^n$ and $w_0 \in \mathbb{Q}$, define

$$\text{init}_{(w_0,\boldsymbol{w})}(f) = \text{sum of terms of } f \text{ with maximal } (w_0, \boldsymbol{w})\text{-degree}$$

Then:

$$V_{\text{trop}}(I) = \{\boldsymbol{w} \in \mathbb{Q}^n : \text{init}_{(1,\boldsymbol{w})}(I) = \langle \text{init}_{(1,\boldsymbol{w})}(f) : f \in I\rangle \text{ does not contain a monomial}\}$$

Computing tropical varieties (1)

"Easy" case: the coefficients do not have a valuation

Naive algorithm:

1. Compute the Gröbner fan of the ideal
2. For each cone:
   2.1 Pick a vector $\boldsymbol{w}$ in the cone, define the corresponding monomial order
   2.2 Compute $\mathrm{init}_{\boldsymbol{w}}(I)$
   2.3 Compute $(\mathrm{init}_{\boldsymbol{w}}(I) : (x_1 \cdots x_n)^\infty)$
   2.4 If the result is 1, add the cone to the tropical variety

Less naive algorithm: do the same thing, but compute the tropical variety as you go, without traversing the entire Gröbner fan.

Hard case 1: $K = \mathbb{Q}((t))$
Compute the Gröbner fan using standard bases and Mora's algorithm

Hard case 2: $K = \mathbb{Q}_p$
Reduce to the previous case by:

1. replacing numbers $\sum a_i p^i$ with series $\sum a_i t^i$
2. adding the equation $p - t$ to the ideal
3. saturating by p

This work: deal with both cases uniformly using Tate series:
instead of working with polynomials over a valued field, we work with convergent series

- Computing the Gröbner fan
- Computing the initial ideal
- Computing the saturation

$I \subset \mathbb{Q}[\boldsymbol{X}]$ homogeneous ideal (without coefficient valuation)

Gröbner cone

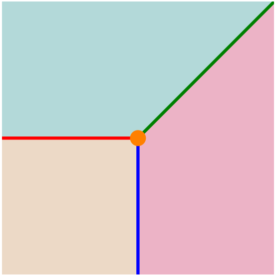
- Equivalence relation on $\mathbb{Q}^n$: $\boldsymbol{a} \sim \boldsymbol{b} \iff \mathrm{init}_{\boldsymbol{a}}(I) = \mathrm{init}_{\boldsymbol{b}}(I)$
- Equivalence classes are (open) rational polyhedral cones
- Cones with maximal dimension correspond to term orders
- There are only finitely many cones

Gröbner fan:

- Subdivision of $\mathbb{Q}^n$ as the union of Gröbner cones
- Lower dimensional cones are boundaries (collisions between leading terms)
- The tropical variety is contained in those lower-dimensional cones

Ex: $f = x^2 + y^2 + 1$

$\boldsymbol{w}$	$\mathrm{init}_{\boldsymbol{w}}(f)$
$(1,0)$	x^2
$(0,1)$	y^2
$(-1,-1)$	1
$(1,1)$	$x^2 + y^2$
$(0,-1)$	$x^2 + 1$
$(-1,0)$	$y^2 + 1$
$(0,0)$	$x^2 + y^2 + 1$



Term orders and universal Gröbner bases

Why are there finitely many equivalence classes of term orders?

Fact 1: given a finite set of polynomials, there are finitely many possible leading terms.

Fact 2: a homogeneous polynomial ideal admits a finite Gröbner basis working for all orders (universal GB)

Both facts are effective for polynomials:

- the possible leading terms are the vertices of the Newton polytope
- the corresponding term orders can be computed
- if there is an order for which the set is not a Gröbner basis, we can compute a Gröbner basis, take the union and repeat
- the process terminates by a Noetherianity argument

Why homogeneous?
This only works for global orders, but we never need to compare terms with different degrees

Tate series

Tate series: convergent series with coefficients in a valued field or ring (e.g. $\mathbb{Q}((T))$)

$$\sum_{\alpha\in\mathbb{N}^n} a_\alpha \boldsymbol{X}^\alpha \text{ with } \mathrm{val}(a_\alpha) - \alpha\cdot\boldsymbol{r} \xrightarrow{|\alpha|\to\infty} \infty$$

- $\boldsymbol{r} = (r_1,\dots,r_n)$: convergence radius
- Notation: $K\{\boldsymbol{X};\boldsymbol{r}\}$
- If $\boldsymbol{r} = (0,\dots,0)$, equivalent: $a_\alpha \to 0$
- If $\boldsymbol{r} \in \mathbb{Z}^n$, equivalent: $\frac{a_\alpha}{T^{|\boldsymbol{r}|}} \to 0$ (coefficients after evaluating at $(X_1/T^{r_1},\dots,X_n/T^{r_n})$)
- $K[\boldsymbol{X}] \subseteq K\{\boldsymbol{X};\boldsymbol{r}\}$ for all $\boldsymbol{r}$ (“infinite” convergence radius)
- If $\boldsymbol{r} \le \boldsymbol{s}$, $K\{\boldsymbol{X};\boldsymbol{s}\} \subseteq K\{\boldsymbol{X};\boldsymbol{r}\}$
- The minimal Gauss valuation component of f is: $\mathrm{init}_{(-1,\boldsymbol{r})}(f)$

Term ordering:

$$a\boldsymbol{X}^\alpha < b\boldsymbol{X}^\beta \iff \begin{cases}\mathrm{val}(a) - \boldsymbol{r}\cdot\alpha > \mathrm{val}(b) - \boldsymbol{r}\cdot\beta \\ \text{or they are equal and } \boldsymbol{X}^\alpha < \boldsymbol{X}^\beta\end{cases}$$

- Every Tate series has a leading term
- This allows to compute Gröbner bases

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha \boldsymbol{X}^\alpha \text{ with } \mathrm{val}(a_\alpha) - \alpha \cdot \boldsymbol{r} \xrightarrow{|\alpha| \to \infty} \infty$$

- If $\boldsymbol{r} \leq \boldsymbol{s}$, then $K\{\boldsymbol{X}; \boldsymbol{s}\} \subseteq K\{\boldsymbol{X}; \boldsymbol{r}\}$ — Overconvergence
- $K[\boldsymbol{X}] \subseteq K\{\boldsymbol{X}; \boldsymbol{r}\}$ for all $\boldsymbol{r}$ — Infinite convergence radius
- If $I \subseteq K\{\boldsymbol{X}; \boldsymbol{s}\}$ or $I \subseteq K[\boldsymbol{X}]$, then $K\{\boldsymbol{X}; \boldsymbol{r}\}I$ can be bigger than I — Completion
  (Ex: $1 + TX \in \mathbb{Q}(T)[X]$ is invertible in $\mathbb{Q}((T))\{X; 0\}$)

Theorem (Caruso, Vaccon, V. 2022)
Let $\boldsymbol{r} \leq \boldsymbol{s}$, $I \subset K\{\boldsymbol{X}; \boldsymbol{r}\}$ and $I_{\boldsymbol{s}} = IK\{\boldsymbol{X}; \boldsymbol{s}\}$.
Then $I_{\boldsymbol{s}}$ admits a Gröbner basis comprised only of elements of $K\{\boldsymbol{X}; \boldsymbol{r}\}$.
In particular, the completion of a polynomial ideal has a polynomial basis.

Key component: Mora's reduction algorithm

Input: G a Gröbner basis (for a local or mixed order) and f

Output: h and u such that

- uf reduces to h modulo G
- $\mathrm{LT}(u) = 1$ (for us: $u = 1 +$ part with positive valuation, or equivalently, u is invertible)
- u and h live in the same ring as f and the elements of G.

$f \in K\{\boldsymbol{X};\boldsymbol{r}\}$

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \boldsymbol{X}^\alpha \text{ with } \operatorname{val}(a_\alpha) - \alpha \cdot \boldsymbol{r} \xrightarrow{|\alpha| \to \infty} \infty$$

The component of minimal Gauss valuation in f is a equal to $\operatorname{init}_{(-1,\boldsymbol{w})}(f)$, and in particular, the latter is well-defined and a polynomial.

Theorem (Vaccon, V. 2023)
Let $I \subseteq K[\boldsymbol{X}]$, and $\boldsymbol{w} \in \mathbb{Q}^{n+1}$ a system of weights.
Let $\boldsymbol{r} = (-w_1/w_0, \dots, -w_n/w_0)$ and $I_{\boldsymbol{r}} = I\,K\{\boldsymbol{X};\boldsymbol{r}\}$ the corresponding completion, then

$$\operatorname{init}_{\boldsymbol{w}}(I) = \operatorname{init}_{\boldsymbol{w}}(I_{\boldsymbol{r}}) \cap K[\boldsymbol{X}].$$

This is a local result, which translates globally as:

$$V_{\text{trop}}(I) = \bigcup_{\boldsymbol{s} \in \mathbb{Q}^n} V_{\text{trop}}(I_{\boldsymbol{s}}).$$

Theorem (Vaccon, V. 2023)
Let G be a Gröbner basis of $I_{\boldsymbol{r}}$, then

$$\operatorname{init}_{\boldsymbol{w}}(I_{\boldsymbol{r}}) = \langle \operatorname{init}_{\boldsymbol{w}}(g) : g \in G \rangle.$$

Theorem (Caruso, Vaccon, V. 2022; Vaccon, V. 2023)
Let $I \subseteq K[\boldsymbol{X}]$ be a homogeneous ideal.
There exists a finite subset $G \subseteq I$ such that
for all $\boldsymbol{r} \in \mathbb{Q}^n$ (and for all monomial orders), G is a Gröbner basis of $I_{\boldsymbol{r}} = I\,K\{\boldsymbol{X};\boldsymbol{r}\}$.
Furthermore, there exists an algorithm to compute it.

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Why homogeneous? Because we need reduced Gröbner bases.

Question:
Let I be an ideal, $\leq_1$ and $\leq_2$ two orders such that $\mathrm{LT}_{\leq_1}(I) = \mathrm{LT}_{\leq_2}(I)$, and G a Gröbner basis of I for $\leq_1$.
Is G a Gröbner basis of I for $\leq_2$?

Universal analytic Gröbner bases

Theorem (Caruso, Vaccon, V. 2022; Vaccon, V. 2023)
Let $I \subseteq K[\boldsymbol{X}]$ be a homogeneous ideal.
There exists a finite subset $G \subseteq I$ such that
for all $\boldsymbol{r} \in \mathbb{Q}^n$ (and for all monomial orders), G is a Gröbner basis of $I_{\boldsymbol{r}} = I\,K\{\boldsymbol{X};\boldsymbol{r}\}$.
Furthermore, there exists an algorithm to compute it.

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Question:
Let I be an ideal, $\leq_1$ and $\leq_2$ two orders such that $\mathrm{LT}_{\leq_1}(I) = \mathrm{LT}_{\leq_2}(I)$, and G a Gröbner basis of I for $\leq_1$.
Is G a Gröbner basis of I for $\leq_2$?

In the usual polynomial case (with global orders), it is not a problem:
we can assume that the basis is reduced and then it is easy.

But for local or mixed orders, we cannot reduce Gröbner bases:

- recall that Mora's algorithm computes u and h such that $u\,f = \cdots + h$ and $\mathrm{LT}(h) < \mathrm{LT}(f)$
- the additional multiple u allows us to get a terminating reduction for the leading term
- but that is a one-time trick: there is no way to prevent infinite tail reductions!

Homogeneous ideals have a reduced Gröbner basis.

This restriction (reduced GB or homogeneous ideal) is also present in the classical case.

Algorithm:
Input: $I \subseteq K[X]$ homogeneous ideal.
Output: the tropical variety of I, given as a union of rational cones

1. compute a universal analytic Gröbner basis G of I (Mora's reduction algorithm + universal GB algorithmn)
2. get all the term orders up to equivalence for I (Newton polytope)
3. get all the maximal dimensional cones in the Gröbner fan (discrete geometry)
4. compute the rest of the cones (discrete geometry)
5. for each non-maximal cone $\mathcal{C}$, pick a $\boldsymbol{w} = (1, -\boldsymbol{r}) \in C$, then $\mathrm{init}_{\boldsymbol{w}}(G)$ is a Gröbner basis of $\mathrm{init}_{\boldsymbol{w}}(I)$ (theorem about Tate Gröbner bases in $K\{\boldsymbol{X}; \boldsymbol{r}\}$)
6. compute $(\mathrm{init}_{\boldsymbol{w}}(I) : (x_1 \cdots x_n)^\infty)$ (polynomial Tate Gröbner bases)
7. conclude whether $\boldsymbol{w} \in V_{\mathrm{trop}}(I)$ and therefore $\mathcal{C} \subseteq V_{\mathrm{trop}}(I)$.

Going further

What we did:

- Application of Tate series to the computation of tropical varieties
- New point of view on the existing algorithms
- It is likely that the optimized algorithms would translate just as well
- First effective application of tropical analytic geometry
  (tropical geometry on convergent sequences)

Going further:

- The full scope of tropical analytic geometry requires much more specialized convergence condition, e.g. convergence on a polyhedron (for several convergence radii which are not component-wise ordered) or on a corona (allowing Laurent polynomials)
- This is currently very far from what we can hope to reach with our machinery
- The main questions in each case are:
  1. Does there exists a Gröbner basis comprised of elements satisfying the same constraints? Can we compute it?
  2. Is there a minimal set of leading terms? Is there a universal analytic Gröbner basis? Can we compute it?
- For polyhedra, we have an algorithm for the first question (based on yet another variant of Mora's reduction)