On Polynomial Ideals And Overconvergence In Tate Algebras

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ABSTRACT

In this paper, we study ideals spanned by polynomials or overconvergent series in a Tate algebra. With state-of-the-art algorithms for computing Tate Gröbner bases, even if the input is polynomials, the size of the output grows with the required precision, both in terms of the size of the coefficients and the size of the support of the series.

We prove that ideals which are spanned by polynomials admit a Tate Gröbner basis made of polynomials, and we propose an algorithm, leveraging Mora's weak normal form algorithm, for computing it. As a result, the size of the output of this algorithm grows linearly with the precision.

Following the same ideas, we propose an algorithm which computes an overconvergent basis for an ideal spanned by overconvergent series.

Finally, we prove the existence of a universal analytic Gröbner basis for polynomial ideals in Tate algebras, compatible with all convergence radii.

CCS CONCEPTS

• **Computing methodologies → Algebraic algorithms**.

KEYWORDS

Algorithms, Gröbner bases, Tate algebra, Mora's algorithm, Universal Gröbner basis

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1 INTRODUCTION

The study of p-adic geometric objects has taken significant importance in the 20th century, as a crucial component of algebraic number theory. Beyond polynomials and algebraic geometry, Tate developed a theory of p-adic analytic varieties, called rigid geometry. This theory is now central to many developments in number theory. The fundamental underlying algebraic object is Tate algebras, that is, algebras of convergent multivariate power series over a complete discrete valuation field K (for instance $\mathbb{Q}_p$).

In earlier papers, the authors examined those Tate series from a computational point of view, with the hope to develop an algorithmic toolbox on par with what is available for polynomials. The main result was that it is possible to define and compute Gröbner bases of Tate ideals, in a way compatible with the usual theory on polynomials over the residue field (e.g. $\mathbb{F}_p$). We also examined how different algorithms from the polynomial case transfer to Tate settings.

A key property of Tate series is their convergence radius. Tate algebras are parameterized by the convergence radius of their series. If a series is convergent on a certain disk, it is certainly convergent on all disks with smaller radius. This gives a natural embedding of one Tate algebra into another if the convergence radii of the latter are smaller than those of the former. This property is a key feature of rigid geometry. In fact, the canonical embedding of $K[\mathbf{X}]$ into Tate algebras is a particular case of such an embedding, with polynomials seen as series with infinite convergence radius.

Beyond this theoretical interest, recognizing and exploiting such overconvergence properties would help making the algorithms more efficient. Indeed, a limiting factor of the current algorithms is the cost of the reductions, in particular as the precision grows. Series with a larger convergence radius are series which converge faster, and thus require to compute fewer terms while reducing. The challenge in taking advantage of those properties lies in designing algorithms ensuring that this overconvergence property is preserved in the course of the algorithm.

In [6], we showed how to generalize the FGLM algorithm [11] to Tate algebras. A result was that this algorithm allows, for zero-dimensional Tate ideals embedded into a Tate algebra with a less restrictive convergence condition, to convert the Gröbner basis. This opens the possibility, for zero-dimensional ideals, of computing a Gröbner basis in the smaller Tate algebra, where all series have the stronger convergence property, and then using FGLM to convert the result.

In this work, we consider ideals spanned by polynomials in a Tate algebra, from this point of view of overconvergence. We show that in this case, the ideal admits a Gröbner basis comprised only of polynomials, and we propose an algorithm computing such a basis, and working only with polynomials. The key ingredient is to use a variant of Mora's weak normal form [15] to compute the head reductions instead of standard reduction. This algorithm computes reductions up to an invertible factor, with the additional property that all series appearing in the computations are actually polynomials. In order to do so, it uses specific metrics, called *écarts*, to select the polynomials to use for reduction at each step. This notion

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of écart is crucial in proving that the Gröbner basis computation terminates. In the polynomial case, the écart is simply defined as the difference between the degree of the polynomial and that of its leading term, and in the Tate case we need to refine that with a comparison on the set-difference of the supports of the polynomials.

The resulting algorithm offers a better control for the complexity as a function of the precision. Concretely, given a set of polynomials in $\mathbb{Q}[\mathbf{X}]$, and a prime number p, we consider the ideal I spanned by the polynomials in $\mathbb{Q}_p\{\mathbf{X}\}$. Using existing algorithms for Tate Gröbner bases, we can compute a Gröbner basis of I modulo p^N for all N. But the output of such a computation will be truncated series, and if we increase the precision N, the size of their support typically increases, and even quantifying that growth is not an easy task. By contrast, the algorithm which we present here only computes polynomials. So once the precision is large enough, the supports will be completely determined and will not grow anymore. Asymptotically, the size of the output grows linearly with the precision, and the complexity of the algorithm grows at the same rate as the cost of coefficient arithmetic.

The same idea can be used for ideals spanned by overconvergent series. With a further refinement of the écart in order to take the valuation of the coefficients into account, we prove that Mora's algorithm allows to compute overconvergent remainders as a result of reducing overconvergent series, and that the modified version of Buchberger's algorithm converges, and computes a basis comprised of overconvergent series.

In later sections, we examine an application of those results for ideals spanned by polynomials in a Tate algebra, namely eliminating variables. This operation is fundamental in effective algebraic geometry, by allowing to compute various ideal operations such as saturation or intersection. However, in the Tate setting, due to the nature of the term ordering, computing an elimination ideal by a Gröbner basis may fail. Concretely, even with an elimination ordering, the leading term of a Tate series is determined by first looking at the valuations of the coefficients, and so it is not enough to look at the leading term to determine whether the series involves the elimination variable or not.

We prove that for ideals spanned by polynomials, using Buchberger's algorithm with Mora reductions, this problem does not appear, and we are indeed able to eliminate variables. This allows to recover all the usual ideal operations, and in particular proves that polynomial ideals are stable under intersection and radical.

Finally, we consider the theory of universal Gröbner bases, that is, sets which are a Gröbner basis for all monomial orderings. This theory has proved useful in the classical setting, for instance leading to algorithms for change of ordering. The key result is that any ideal in a polynomial algebra has a finite universal Gröbner basis. This allows to see the set of Gröbner bases of the ideal as a polyhedral cone, and algorithms wandering on this cone have been developed (see [1, 8, 12, 16]). Furthermore, connections with tropical geometry have been explored (see [2, 14]).

This latter aspect motivates the quest for a similar notion in the Tate setting. It could pave the way for the computation of the tropical analytic variety defined by a polynomial ideal (see [17]). However, it is not clear whether all Tate ideals admit a finite universal Gröbner basis. The last result of this work is a proof that polynomial ideals admit a finite universal analytic Gröbner basis, valid regardless of the choice of the convergence radii.

2 SETTING

2.1 Term orders, Tate algebras and ideals

In order to fix notations, we briefly recall the definition of Tate algebras and their theory of Gröbner bases (GB for short). Let K be a field with valuation val and let K° be the subring of K consisting of elements of nonnegative valuation. Let π be a uniformizer of K, that is an element of valuation 1. Let $K^{\mathrm{ex}} \subset K$, be an exact field. Typical examples of such a setting are p-adic fields like $K = \mathbb{Q}_p$ with $K^\circ = \mathbb{Z}_p$, $\pi = p$ and $K^{\mathrm{ex}} = \mathbb{Q}$ or Laurent series fields like $K = \mathbb{Q}((T))$ with $K^\circ = \mathbb{Q}[[T]]$, $\pi = T$ and $K^{\mathrm{ex}} = \mathbb{Q}(T)$.

Let $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{Q}^n$. The *Tate algebra* $K\{\mathbf{X}; \mathbf{r}\}$ is defined by:

$$K\{\mathbf{X};\mathbf{r}\} := \left\{ \sum_{\mathbf{i}\in\mathbb{N}^n} a_{\mathbf{i}}\mathbf{X}^{\mathbf{i}} \text{ s.t. } a_{\mathbf{i}} \in K \text{ and } \mathrm{val}(a_{\mathbf{i}}) - \mathbf{r}\cdot\mathbf{i} \xrightarrow[|\mathbf{i}|\to+\infty]{} +\infty \right\} \tag{1}$$

The tuple $\mathbf{r}$ is called the convergence log-radii of the Tate algebra. We define the Gauss valuation of a term $a_{\mathbf{i}}\mathbf{X}^{\mathbf{i}}$ as $\mathrm{val}_{\mathbf{r}}(a_{\mathbf{i}}\mathbf{X}^{\mathbf{i}}) = \mathrm{val}(a_{\mathbf{i}}) - \mathbf{r}\cdot\mathbf{i}$, and the Gauss valuation of $\sum a_{\mathbf{i}}\mathbf{X}^{\mathbf{i}} \in K\{\mathbf{X};\mathbf{r}\}$ as the minimum of the Gauss valuations of its terms. The integral Tate algebra ring $K\{\mathbf{X};\mathbf{r}\}^\circ$ is the subring of $K\{\mathbf{X};\mathbf{r}\}$ consisting of elements with nonnegative valuation.

We fix once and for all a classical *monomial order* $\leq_m$ on the set of monomials $\mathbf{X}^{\mathbf{i}}$. Given two terms $a\mathbf{X}^{\mathbf{i}}$ and $b\mathbf{X}^{\mathbf{j}}$ (with $a, b \in K^\times$), we write $a\mathbf{X}^{\mathbf{i}} <_{\mathbf{r}} b\mathbf{X}^{\mathbf{j}}$ if $\mathrm{val}_{\mathbf{r}}(a\mathbf{X}^{\mathbf{i}}) > \mathrm{val}_{\mathbf{r}}(b\mathbf{X}^{\mathbf{j}})$, or $\mathrm{val}(a\mathbf{X}^{\mathbf{i}}) = \mathrm{val}(b\mathbf{X}^{\mathbf{j}})$ and $\mathbf{X}^{\mathbf{i}} <_m \mathbf{X}^{\mathbf{j}}$. The leading term of a Tate series $f = \sum a_{\mathbf{i}}\mathbf{X}^{\mathbf{i}} \in K\{\mathbf{X};\mathbf{r}\}$ is, by definition, its maximal term, and is denoted by $\mathrm{LT}_{\mathbf{r}}(f)$. Its coefficient and its monomial are denoted $\mathrm{LC}_{\mathbf{r}}(f)$ and $\mathrm{LM}_{\mathbf{r}}(f)$, with $\mathrm{LT}_{\mathbf{r}}(f) = \mathrm{LC}_{\mathbf{r}}(f) \times \mathrm{LM}_{\mathbf{r}}(f)$. For $f, g \in K\{\mathbf{X};\mathbf{r}\}$, we define their S-polynomial as

$$\text{S-Poly}(f,g) = \frac{\mathrm{LT}_{\mathbf{r}}(g)}{\gcd(LT(f), \mathrm{LT}_{\mathbf{r}}(g))} f - \frac{\mathrm{LT}_{\mathbf{r}}(f)}{\gcd(\mathrm{LT}_{\mathbf{r}}(f), \mathrm{LT}_{\mathbf{r}}(g))} g.$$

A Gröbner basis (or GB for short) of an ideal I of $K\{\mathbf{X};\mathbf{r}\}$ is, by definition, a family $(g_1, \ldots, g_s)$ of elements of I with the property that for all $f \in I$, there exists an index $i \in \{1, \ldots, s\}$ such that $\mathrm{LT}_{\mathbf{r}}(g_i)$ divides $\mathrm{LT}_{\mathbf{r}}(f)$. The following theorem is proved in [4].

Theorem 2.1. *Any ideal of $K\{\mathbf{X};\mathbf{r}\}$ or $K\{\mathbf{X};\mathbf{r}\}^\circ$ admits a Gröbner basis.*

We define the monoid of terms $T\{\mathbf{X};\mathbf{r}\}$ as the multiplicative monoid consisting of the elements $a\mathbf{X}^\alpha$ with $a \in K^\times$ and $\alpha \in \mathbb{N}^n$. We let also $T\{\mathbf{X};\mathbf{r}\}^\circ$ be the submonoid of $T\{\mathbf{X};\mathbf{r}\}$ consisting of terms $a\mathbf{X}^{\mathbf{i}}$ for which $\mathrm{val}_{\mathbf{r}}(a\mathbf{X}^{\mathbf{i}}) \geq 0$. The multiplicative group $K^\times$ (resp. $(K^\circ)^\times$) embeds into $T\{\mathbf{X};\mathbf{r}\}$ (resp. $T\{\mathbf{X};\mathbf{r}\}^\circ$). We set:

$$\mathbb{T}\{\mathbf{X};\mathbf{r}\} = T\{\mathbf{X};\mathbf{r}\}/K^\times \quad \text{and} \quad \mathbb{T}\{\mathbf{X};\mathbf{r}\}^\circ = T\{\mathbf{X};\mathbf{r}\}^\circ/(K^\circ)^\times.$$

We remark that G is a GB of an ideal I in $K\{\mathbf{X};\mathbf{r}\}$ (resp. $K\{\mathbf{X};\mathbf{r}\}^\circ$) if and only if $\mathrm{LT}_{\mathbf{r}}(G)$ generates $\mathrm{LT}_{\mathbf{r}}(I)$ in $\mathbb{T}\{\mathbf{X};\mathbf{r}\}$ (resp. $\mathbb{T}\{\mathbf{X};\mathbf{r}\}^\circ$).

2.2 Polynomial and overconvergent ideals

The main object of our studies is polynomials and overconvergent series, and the ideals they span.

Definition 2.2. An ideal of $K\{\mathbf{X};\mathbf{r}\}$ is called a *polynomial ideal* if it is spanned by polynomials in $K[\mathbf{X}]$.

Let $\mathbf{s} \leq \mathbf{r}$ with respect to component-wise comparison: $\forall i \in [\![1,n]\!], s_i \leq r_i$. A series $f = \sum_i a_\mathbf{i} \mathbf{X}^\mathbf{i} \in K\{\mathbf{X};\mathbf{r}\}$ is called $\mathbf{s}$-*convergent* (or simply *overconvergent* if $\mathbf{s}$ is clear by the context) if

$$\mathrm{val}(a_\mathbf{i}) - \mathbf{s} \cdot \mathbf{i} \rightarrow_{|i| \rightarrow \infty} +\infty. \tag{2}$$

Equivalently, it means that f is the image of an element of $K\{\mathbf{X};\mathbf{s}\}$ under the canonical embedding. An ideal of $K\{\mathbf{X};\mathbf{r}\}$ is called $\mathbf{s}$-*convergent* if it is spanned by $\mathbf{s}$-convergent series.

Remark 2.3. A polynomial ideal in $K\{\mathbf{X};\mathbf{r}\}$ contains more polynomials than the ideal taken in $K[\mathbf{X}]$. For example, let $\mathbf{r} = (0)$ and consider $f = X + pX^2$ in $\mathbb{Q}_p[X]$. In $\mathbb{Q}_p[X]$, the ideal spanned by f is $\langle f \rangle = \langle X + pX^2 \rangle$.

On the other hand, in $\mathbb{Q}_p\{X\}$, $1 + pX$ is invertible with inverse $\sum_i (-p)^i X^i$, and the ideal contains $X = \left(\sum_i (-p)^i X^i\right) f$.

The following structural results are immediate.

Proposition 2.4. *Let $\mathbf{s} \leq \mathbf{r}$. Let I and J be two polynomial (resp. $\mathbf{s}$-convergent) ideals in $K\{\mathbf{X};\mathbf{r}\}$. Then:*

(1) *the sum $I + J$ is a polynomial (resp. $\mathbf{s}$-convergent) ideal;*
(2) *the product IJ is a polynomial (resp. $\mathbf{s}$-convergent) ideal.*

On the other hand, closure under elimination is not obvious, and therefore closure under intersection or saturation is not immediate. For intersection, it can be proved using that, as a completion of a Noetherian ring, $K\{\mathbf{X};\mathbf{r}\}$ is flat over $K\{\mathbf{X};\mathbf{s}\}$. Using Gröbner bases, we present in Section 3.2 a constructive proof for polynomial ideals.

3 POLYNOMIAL IDEALS: TOOLS AND MOTIVATIONS

Using elimination, we motivate our results with the closure of polynomial ideals under some ideal operations including intersection and saturation.

3.1 Elimination of one variable

Let $A = K\{x_1, \ldots, x_n; r_1, \ldots, r_n\}$ be a Tate algebra with tie-breaking monomial ordering $\leq_{m_A}$. Let $B = K\{t, x_1, \ldots, x_n; r_0, r_1, \ldots, r_n\}$ be a Tate algebra above A. Let I be an ideal of B. We would like to compute a GB of the ideal $I \cap A$ in A (for the monomial ordering $\leq_{m_A}$).

Proposition 3.1. *If $r_0 = +\infty$ and $\leq_{m_A}$ is a block-monomial ordering with t bigger than any monomial not involving t, if G_B is a GB of I for the term ordering defined by the r_i's and $\leq_{m_A}$, then $G_B \cap A$ is a GB of $I \cap A$.*

Proof. Firstly, $\langle G_B \cap A \rangle_A \subset I \cap A$. Now, let us remark that if $g \in B$ is such that $\mathrm{LT}(g) \in A$ then $g \in A$. Indeed, as $r_0 = +\infty$ and $\leq_{m_A}$ is a block-monomial ordering, any term cx^α involving t is such that $t > \mathrm{LT}(g)$ so g does not have any term involving t.

As a consequence of this fact, if $f \in I \cap A$ is divided by G_B then only elements of G_B in A, with LT's in A will be involved, and as G_B is a GB of I, f is reduced to 0. Consequently, the same division to 0 happens for the division of f by $G_B \cap A$, so $I \cap A \subset \langle G_B \cap A \rangle_A$, which concludes the proof. □

Corollary 3.2. *For r_0 big enough, the previous result for $G_B \cap A$ is preserved, allowing to fit into the framework of algorithms developped in [4–6] and in this article.*

3.2 Application to ideal operations

Following §4 of [9], if I and J are ideals then GBs of $I \cap J$, $I : J$ and $I : J^\infty$ can be computed using elimination (*e.g.* $I \cap J = (tI + (1-t)J) \cap K\{\mathbf{X}\}$).

One motivation for our work is Corollary 5.4, stating that any polynomial ideal in $K\{\mathbf{X};\mathbf{r}\}$ admits a GB made of polynomials. It implies the following stability result on polynomial ideals: if I and J are ideals in $K\{\mathbf{X};\mathbf{r}\}$ generated by polynomials, then so are: $I + J$, IJ, $I \cap J$, $I : J$ and $I : J^\infty$.

3.3 Homogenization and dehomogenization

We will rely on (de)-homogenization at some point in the computations. We consign here notations and basic properties.

Definition 3.3. Let $(\cdot)^*$ and $(\cdot)_*$ be the homogenization and dehomogenization applications between $K[\mathbf{X}]$ and $K[\mathbf{X}, t]$. If I is an ideal in $K[\mathbf{X}]$, we define I^* to be the homogenization of this ideal in $K[\mathbf{X}, t]$.

Given $\mathbf{r} \in \mathbb{Q}^n$, we extend the term order $<_\mathbf{r}$ to $K[\mathbf{X}, t]$ and $K\{\mathbf{X}, t; \mathbf{r}, 0\}$

Definition 3.4. Given two terms $a\mathbf{X}^\alpha t^u$ and $bX^\beta t^v$, we write that $a\mathbf{X}^\alpha t^u <_{\mathbf{r},0} bX^\beta t^v$ if:

- $\mathrm{val}_\mathbf{r}(a\mathbf{X}^\mathbf{i}) > \mathrm{val}_\mathbf{r}(b\mathbf{X}^\mathbf{j})$ (which is the same as $\mathrm{val}_{\mathbf{r},0}(a\mathbf{X}^\alpha t^u) > \mathrm{val}_{\mathbf{r},0}(bX^\beta t^v)$).
- $\mathrm{val}_\mathbf{r}(a\mathbf{X}^\mathbf{i}) = \mathrm{val}_\mathbf{r}(b\mathbf{X}^\mathbf{j})$ and $\deg(\mathbf{X}^\alpha t^u) < \deg(X^\beta t^v)$.
- $\mathrm{val}_\mathbf{r}(a\mathbf{X}^\mathbf{i}) = \mathrm{val}_\mathbf{r}(b\mathbf{X}^\mathbf{j})$, $\deg(\mathbf{X}^\alpha t^u) = \deg(X^\beta t^v)$ and $\mathbf{X}^\alpha <_m \mathbf{X}^\beta$.

This defines a term order on $K\{\mathbf{X}, t; \mathbf{r}, 0\}$.

This order is defined such that dehomogenization preserves leading terms of homogeneous polynomials of $K[\mathbf{X}, t]$.

Lemma 3.5. *Let $\mathbf{r} \in \mathbb{Q}^n$. Let $h \in K[\mathbf{X}, t]$ be a homogeneous polynomial. Then $\mathrm{LT}_{(\mathbf{r},0)}(h)_* = \mathrm{LT}_\mathbf{r}(h_*)$. Let $f \in K[\mathbf{X}]$, then $\mathrm{LT}_\mathbf{r}(f) = (\mathrm{LT}_{(\mathbf{r},0)}(f^*))_*$.*

Proof. Thanks to the way we defined the term order on $K\{\mathbf{X}, t; \mathbf{r}, 0\}$ in Definition 3.4, if $c_\alpha x^\alpha t^u$ and $c_\beta x^\beta t^v$ are two terms of the same total degree such that $c_\alpha x^\alpha t^u >_{\mathbf{r},0} c_\beta x^\beta t^v$, then $c_\alpha x^\alpha >_\mathbf{r} c_\beta x^\beta$. This is enough for the first part. For the second part, we can conclude using $h = f^*$ and the fact that $f = (f^*)_*$. □

4 WEAK NORMAL FORMS

4.1 Definitions

We present here how to adapt Mora's tangent cone algorithm to compute Weak Normal Forms over Tate algebras. The main consequence of this notion is that it will allow us, if the generating Tate series are polynomials, to do all computations on polynomials, avoiding any infinite division.

In this section, we fix some $\mathbf{r} \in \mathbb{Q}^n$. First, we recall the definition of weak normal forms, adapted to the framework of polynomial ideals in Tate algebras.

Definition 4.1. A *weak normal form* is a map $\mathsf{WNF} : K[\mathbf{X}] \times \mathcal{P}(K[\mathbf{X}]) \to K[\mathbf{X}]$, such that, for all $f \in K[\mathbf{X}]$ and all $G \subseteq K[\mathbf{X}]$, the following holds:

(1) $\mathsf{WNF}(0, G) = 0$
(2) If $\mathsf{WNF}(f, G) \neq 0$, then $\mathrm{LT}(\mathsf{WNF}(f, G))$ does not lie in the ideal spanned by the leading terms of G
(3) If $f \neq 0$, then there exists $u \in K[\mathbf{X}]$ invertible in $K\{\mathbf{X}; \mathbf{r}\}$ such that $uf - \mathsf{WNF}(f, G) = \sum_{g \in G} u_g g$ with the u_g's polynomials, $\mathrm{LT}_{\mathbf{r}}(u_g g) \leq \mathrm{LT}_{\mathbf{r}}(f)$ with equality attained at most once.[1]

In particular, if $\mathsf{WNF}(f, G) = 0$, then f lies in the ideal spanned by G. And if G is a Gröbner basis, it is an equivalence.

4.2 Écarts

The first step in order to devise a new version of Mora's tangent cone algorithm is to provide a suitable écart function on polynomials. This function then drives the division algorithm. To do so, we adapt the écart functions from [15] and [7] to fit into the Tate algebra framework (see also [13] for a general background on standard bases computations).

Definition 4.2. For f a polynomial, we define:

$$\text{Écart}_1(f) := \deg(f) - \deg(\mathrm{LM}_{\mathbf{r}}(f)).$$

Definition 4.3. For $h = \sum_u b_u x^u$ and $g = \sum_u c_u x^u$ two polynomials, we define:

$$\text{Écart}_2(h, g) := \mathrm{card}(\{u : b_u = 0, c_u \neq 0\}).$$

4.3 Mora's Weak Normal Form algorithm

We first present a simple version of Mora's algorithm to compute a Weak Normal Form (WNF) of a polynomial modulo a finite set of polynomials. It differs from the multivariate division algorithm by adding intermediate reduced polynomials to the list of divisors, which induces a division which happens, not on the original divided polynomial, but on one of his multiples by an invertible polynomial (which does not modify the $\mathrm{LT}_{\mathbf{r}}$'s).

Algorithm 1 $\mathrm{WNF}(f, g)$, Mora's Weak Normal Form algorithm

Input: $f, g_1, \ldots, g_s \in K[\mathbf{X}]$
Output: $h \in K[\mathbf{X}]$ such that for some $\mu, u_1, \ldots, u_s \in K[\mathbf{X}]$, $\mu f = \sum u_i g_i + r$,
μ is a polynomial such that $\mathrm{val}_{\mathbf{r}}(\mu - 1) > 0$, and when $h \neq 0$, $\mathrm{LT}_{\mathbf{r}}(h)$ is divisible by no $\mathrm{LT}_{\mathbf{r}}(g_i)$'s.
Moreover, $\mathrm{LT}_{\mathbf{r}}(u_i g_i) \leq \mathrm{LT}_{\mathbf{r}}(f)$.

1: $h := f$;
2: $T := (g_1, \ldots, g_s)$;
3: **while** $h \neq 0$ and $T_h := \{g \in T, \mathrm{LT}_{\mathbf{r}}(g) \mid \mathrm{LT}_{\mathbf{r}}(h)\} \neq \emptyset$ **do**
4: choose $g \in T_h$ minimizing first $\text{Écart}_1(g)$ then $\text{Écart}_2\left(h, \frac{\mathrm{LM}_{\mathbf{r}}(h)}{\mathrm{LM}_{\mathbf{r}}(g)} g\right)$;
5: **if** $\text{Écart}_1(g) > \text{Écart}_1(h)$, or $\text{Écart}_2(h, \frac{\mathrm{LM}_{\mathbf{r}}(h)}{\mathrm{LM}_{\mathbf{r}}(g)} g) > 0$ **then**
6: $T := T \cup \{h\}$;
7: $h := \text{S-Poly}(h, g)$;
8: **return** h ;

[1] This is sometimes called a *strong Gröbner representation* of f by G.

We may remark that if $\mu \in K[\mathbf{X}]$ is such that $\mathrm{val}_{\mathbf{r}}(\mu - 1) > 0$, then μ is invertible in $K\{\mathbf{X}; \mathbf{r}\}$.

4.4 Termination

Lemma 4.4. *Algorithm 1 terminates.*

Proof. Let us define the extended leading terms (with respect to Écart_1) as: $\mathrm{LTE} : K[\mathbf{X}] \to K[\mathbf{X}, t]$ with $LTE(f) = \mathrm{LT}_{\mathbf{r}}(f) \times t^{\text{Écart}_1(f)}$.

Let us assume the algorithm does not terminate. It means that T_h is never empty. From Prop 2.8 of [4], there exists some N such that $\mathrm{LTE}(T^{(v)})$ is stable for $v \geq N$.

For $v \geq N$, when h_v is processed, two possibilities can occur. If it is not added to T on Line 6, it means that for the selected reducer g, $\text{Écart}_1(g) \leq \text{Écart}_1(h_v)$. If it is added, then $\mathrm{LTE}(h_v)$ is in $\mathrm{LTE}(T^{(v)})$ and hence, there is some $g \in T^{(v)}$ such that $\mathrm{LTE}(g) \mid \mathrm{LTE}(h_v)$. It means that $\text{Écart}_1(g) \leq \text{Écart}_1(h_v)$ and $\mathrm{LT}_{\mathbf{r}}(g) \mid \mathrm{LT}_{\mathbf{r}}(h_v)$.

Thus, in both cases, the g selected in Line 4 has to be such that $\text{Écart}_1(g) \leq \text{Écart}_1(h_v)$. In consequence, starting from $v \geq N$, $\deg(h_v)$ can not increase, and is upper-bounded by d.

Thereafter, the amount of $\mathrm{LM}_{\mathbf{r}}(h_j)$'s and $x^\alpha \mathrm{LM}_{\mathbf{r}}(g)$'s for $g \in T$ and $\deg(x^\alpha \mathrm{LM}_{\mathbf{r}}(g)) \leq d$ is finite. Moreover, for the polynomials reaching such an $\mathrm{LM}_{\mathbf{r}}$, only a finite amount of supports are possible.

Therefore, there is some $N_2 > N$ such that after the N_2-th term, T will not gain any new support for its polynomials nor their monomial multiples of degree $\leq d$. Then, for $v \geq N_2$, the minimal Écart_2 is 0. Indeed, if it is not 0, then h_v is added to T. But as $v \geq N_2$, there is a g with $\mathrm{LT}_{\mathbf{r}}(g) \mid \mathrm{LT}_{\mathbf{r}}(h_v)$ and $Support(\frac{\mathrm{LM}_{\mathbf{r}}(h)}{\mathrm{LM}_{\mathbf{r}}(g)} g) = Support(h_v)$, and thus $\text{Écart}_2(\frac{\mathrm{LM}_{\mathbf{r}}(h)}{\mathrm{LM}_{\mathbf{r}}(g)} g, h_v) = 0$, which is a contradiction.

Hence, for $m \geq N_2$ necessarily, it means that $\mathrm{Supp}(h_{m+1}) \subsetneq \mathrm{Supp}(h_m)$ (the leading term of h_m being canceled). Since the size of the support cannot decrease indefinitely, the algorithm must terminate. □

4.5 Correctness

In order to prove correctness, we extend the algorithm so that the production of the cofactors is explicit (Algorithm 2).

Correctness then comes from the following loop invariant:

Lemma 4.5. *For any $j \geq 0$,*

(1) $\mu_j f = h_j + \sum_i u_{i,j} g_i$,
(2) $\mathrm{val}_{\mathbf{r}}(\mu_j - 1) > 0$,
(3) $\mathrm{LT}_{\mathbf{r}}(u_{i,j} g_i) \leq \mathrm{LT}_{\mathbf{r}}(f)$, *with equality attained at most once, and if so, always with the same i for all j;*
(4) $\mathrm{LT}_{\mathbf{r}}(h_{j+1}) < \mathrm{LT}_{\mathbf{r}}(h_j)$.

Proof. It is clearly true when entering the first loop.

The equality for the third item is attained once after the end of the first loop.

Inside the loop, there is no difficulty when the reduction is performed by one of the initial g_i's. One applies the fourth item to ensure that no second $\mathrm{LT}_{\mathbf{r}}(u_{i,j} g_i)$ reaches $\mathrm{LT}_{\mathbf{r}}(f)$.

When the divisor g was added to T at a previous iteration of the algorithm, *i.e.* $g = h_m$ for some $m < j$, then the situation is the following. Firstly, the preservation of the fourth item is clear.

Then, as $m < j$, we get from the fourth item of the loop invariant that $\mathrm{LT}_{\mathbf{r}}(h_j) < \mathrm{LT}\,\mathbf{r}(g)$ and also $\mathrm{LT}_{\mathbf{r}}(c_v x^v g) = \mathrm{LT}_{\mathbf{r}}(h_j)$. It implies

Algorithm 2 Mora's Weak Normal Form algorithm with cofactors

Input: $f, g_1, \dots, g_s \in K[\mathbf{X}]$

Output: $\mu, u_1, \dots, u_s, h \in K[\mathbf{X}]$ such that $\mu f = \sum u_i g_i + h$ when $h \neq 0$, $\mathrm{LT}_{\mathbf{r}}(h)$ is divisible by no $\mathrm{LT}_{\mathbf{r}}(g_i)$'s and μ is a polynomial such that $\mathrm{val}_{\mathbf{r}}(\mu - 1) > 0$.

1: $h_0 := f, \mu_0 = 1, u_{1,0} = \cdots = u_{s,0} = 0, j = 0$;
2: $T := (g_1, \dots, g_s)$;
3: **while** $h_j \neq 0$ and $T_{h_j} := \{g \in T, \mathrm{LT}_{\mathbf{r}}(g) \mid \mathrm{LT}_{\mathbf{r}}(h_j)\} \neq \emptyset$ **do**
4: choose $g \in T_{h_j}$ minimizing first $\text{Écart}_1(g)$ then $\text{Écart}_2\left(h_j, \frac{\mathrm{LM}_{\mathbf{r}}(h_j)}{\mathrm{LM}_{\mathbf{r}}(g)} g\right)$;
5: **if** $\text{Écart}_1(g) > \text{Écart}_1(h)$, or $\text{Écart}_2(h_j, \frac{\mathrm{LM}_{\mathbf{r}}(h)}{\mathrm{LM}_{\mathbf{r}}(g)} g) > 0$ **then**
6: $T := T \cup \{h_j\}$;
7: $x^v := \mathrm{LM}_{\mathbf{r}}(h_j)/\mathrm{LM}_{\mathbf{r}}(g)$, $c_v := LC_{\mathbf{r}}(h_j)/LC_{\mathbf{r}}(g)$;
8: $h_{j+1} :=$ S-Poly(h_j, g) *i.e.* $h_{j+1} := h_j - c_v x^v g$;
9: **if** $g = g_m$ for some $1 \leq m \leq s$ **then**
10: $u_{m,j+1} := u_{m,j} + c_v x^v, u_{i,j+1} = u_{i,j}$ for $i \neq m$, $\mu_{j+1} := \mu_j$;
11: **else**
12: g was added to T at some previous iteration of the algorithm, so $g = h_m$ for some $m < j$;
13: $\mu_{j+1} := \mu_j - c_v x^v \mu_m$, and for all $i \leq s$, $u_{i,j+1} := u_{i,j} - c_v x^v u_{i,m}$;
14: $j := j + 1$;
15: **return** $\mu_j, u_{1,j}, \dots, u_{s,j}, h_j$;

that $\mathrm{val}_{\mathbf{r}}(c_v x^v) > 0$. Hence, as $\mathrm{val}_{\mathbf{r}}(\mu_j - 1) > 0$, the same is true for $\mu_{j+1} := \mu_j - c_v x^v \mu_m$ and the second item is preserved.

From $\mu_j f = h_j + \sum_i u_{i,j} g_i$, and $\mu_m f = h_m + \sum_i u_{i,m} g_i m$, one gets $(\mu_j - c_v x^v \mu_m) f = (h_j - c_v x^v h_m) + \sum_i (u_{i,j} - c_v x^v u_{i,m}) g_i$ so $\mu_{j+1} f = h_{j+1} + \sum_i u_{i,j+1} g_i + r_{j+1}$, and the first item is preserved.

As $\mathrm{val}_{\mathbf{r}}(c_v x^v) > 0$, then $\mathrm{LT}_{\mathbf{r}}(c_v x^v u_{i,m}) < \mathrm{LT}_{\mathbf{r}}(u_{i,m})$, so

$$\begin{aligned}\mathrm{LT}_{\mathbf{r}}(u_{i,j} - c_v x^v u_{i,m}) &\leq \max(\mathrm{LT}_{\mathbf{r}}(u_{i,j}), \mathrm{LT}_{\mathbf{r}}(c_v x^v u_{i,m})),\\ &\leq \max(\mathrm{LT}_{\mathbf{r}}(u_{i,j}), \mathrm{LT}_{\mathbf{r}}(u_{i,m})),\end{aligned}$$

which is enough to obtain that the third item is preserved and concludes the proof. □

Corollary 4.6. *Algorithm 1 computes a weak normal form.*

Proof. We verify the three items of the definition of weak normal forms. If $f = 0$, the algorithm immediately returns 0.

Assume that $\mathrm{WNF}(f, G) \neq 0$. This implies that after the last loop of the algorithm, $T_h = \emptyset$, and since $\mathrm{LT}_{\mathbf{r}}(T)$ contains the leading terms of G, $\mathrm{LT}_{\mathbf{r}}(\mathrm{WNF}(f, G))$ is not divisible by any of the $\mathrm{LT}_{\mathbf{r}}(G)$.

Finally, the third item follows from the third item of Lemma 4.5. □

Corollary 4.7. *If G, a finite set of polynomials, is a GB of $I_{\mathbf{r}} \subset K\{\mathbf{X}; \mathbf{r}\}$, then for any polynomial $f \in I_{\mathbf{r}}$, $\mathrm{WNF}(f, G) = 0$.*

Proof. If G is a GB of $I_{\mathbf{r}}$, then when dividing $f \in I_{\mathbf{r}}$, on Line 3, T_h is never empty. Indeed, from the first item of the loop invariant, $h_j = \mu_j f - \sum_i u_{i,j} g_i$ means that $h_j \in I_{\mathbf{r}}$. Consequently, the algorithm can only terminate if h reaches 0. □

5 BUCHBERGER'S ALGORITHM WITH WNF

5.1 Description of the algorithm

We prove Buchberger's criterion following the lines of §3.2 of [4]. We rely on a small variation of the technical Lemma 3.6 of [4], which is a generalization of [3, Sec. 2.10, Prop. 5]:

Lemma 5.1. *Let $h_1, \dots, h_m \in K\{\mathbf{X}; \mathbf{r}\}$ and $t_1, \dots, t_m \in T\{\mathbf{X}; \mathbf{r}\}$. We assume that the $\mathrm{LT}_{\mathbf{r}}(t_i h_i)$'s all have the same image in $\mathbb{T}\{\mathbf{X}; \mathbf{r}\}$ and that $\mathrm{LT}_{\mathbf{r}}(\sum_{i=1}^m t_i h_i) < \mathrm{LT}_{\mathbf{r}}(t_1 h_1)$. Then*

$$\sum_{i=1}^{m} t_i h_i = \sum_{i=1}^{m-1} t_i' \cdot \text{S-Poly}(h_i, h_{i+1}) + t_m' \cdot h_m$$

for some $t_1', \dots, t_m' \in T\{\mathbf{X}; \mathbf{r}\}$ such that $\mathrm{LT}_{\mathbf{r}}(t_i' \cdot \text{S-Poly}(h_i, h_{i+1})) < \mathrm{LT}_{\mathbf{r}}(t_1 h_1)$ for $i \in \{1, \dots, m-1\}$ and $\mathrm{LT}_{\mathbf{r}}(t_m' \cdot h_m) < \mathrm{LT}_{\mathbf{r}}(t_1 h_1)$.

Proof. By assumption, there exist $\alpha \in \mathbb{N}^n$ and $d_1, \dots, d_m \in K^\times$ such that $\mathrm{LT}_{\mathbf{r}}(t_i h_i) = d_i \mathbf{X}^\alpha$ for all $i \in \{1, \dots, m\}$. Moreover all the d_i's have the same valuation, say μ. Then $\mathrm{val}(\sum_i d_i) > \mu$. We define $p_i = \frac{t_i h_i}{d_i}$, so that $t_i h_i = d_i p_i$. Then

$$\begin{aligned}\sum_i t_i h_i = {} & d_1(p_1 - p_2) + (d_1 + d_2)(p_2 - p_3) + \cdots + \\ & (d_1 + \cdots + d_{m-1})(p_{m-1} - p_m) + (d_1 + \cdots + d_m) p_m.\end{aligned}$$

Observing that $p_i - p_{i+1} = t_i' \cdot \text{S-Poly}(h_i, h_{i+1})$ with t_i' a term, we get the first t_i''s and $t_m' = \frac{\sum_i d_i}{d_m} t_m$.

Then, since $\mathrm{val}(\sum_i d_i) > \mu$, clearly $\mathrm{LT}_{\mathbf{r}}(t_m' \cdot h_m) < \mathrm{LT}_{\mathbf{r}}(t_1 h_1)$. Moreover, for any $i \in \{1, \dots, m-1\}$, due to the cancellation in S-Poly(h_i, h_{i+1}), $\mathrm{LT}_{\mathbf{r}}(p_i - p_{i+1}) < \mathrm{LT}_{\mathbf{r}}(p_i) = \frac{\mathrm{LT}_{\mathbf{r}}(t_i h_i)}{d_i}$. Considering valuations, $\sum_{k \leq i} d_k \cdot \frac{\mathrm{LT}(t_i h_i)}{d_i} \leq \mathrm{LT}_{\mathbf{r}}(t_i h_i)$. Consequently, $\mathrm{LT}_{\mathbf{r}}(t_i' \cdot \text{S-Poly}(h_i, h_{i+1})) < \mathrm{LT}_{\mathbf{r}}(t_i h_i)$, which concludes the proof. □

Proposition 5.2 (Buchberger's criterion). *G is a GB of $I_{\mathbf{r}}$ if and only if G generates $I_{\mathbf{r}}$ and $\mathrm{WNF}(\text{S-Poly}(g_i, g_j), G) = 0$ for all pairs $g_i, g_j \in G$.*

Proof. The $\Rightarrow$ part is direct thanks to Corollary 4.7.

Let us prove the $\Leftarrow$ part. Let $f \in I$ be such that $\mathrm{LT}_{\mathbf{r}}(f) \notin \langle \mathrm{LT}_{\mathbf{r}}(G) \rangle$. As G generates I, f can be written as $f = \sum_{i=1}^s h_i g_i$ for some Tate series h_i's in $K\{\mathbf{X}; \mathbf{r}\}$.

Let $t = \max_i \mathrm{LT}_{\mathbf{r}}(h_i g_i)$. As $\mathrm{LT}_{\mathbf{r}}(f) \notin \langle \mathrm{LT}_{\mathbf{r}}(G) \rangle$, then $\mathrm{LT}_{\mathbf{r}}(f) < t$. Consequently, among the decompositions of f using G, there is one such that t is minimal.

Let J be the set of indices i such that $\mathrm{LT}_{\mathbf{r}}(h_i g_i) = c_i t$ for some $c_i \in O_K^\times$. Let $t_i = \mathrm{LT}_{\mathbf{r}}(h_i)$ for $i \in J$. Let $h = \sum_{i \in J} t_i g_i$. We have $\mathrm{LT}_{\mathbf{r}}(h) < t$ and $\mathrm{card}(J) \geq 2$ as a cancellation has to appear. We apply Lemma 5.1: there exist terms $t_{j,l}'$ and t' and an index $j_0 \in J$ such that $h = \sum_{j,l \in J} t_{j,l}' \text{S-Poly}(g_j, g_l) + t' g_{j_0}$ and $\mathrm{LT}_{\mathbf{r}}(t' g_{j_0}) < t$ and $\mathrm{LT}_{\mathbf{r}}(t_{j,l}' \text{S-Poly}(g_j, g_l)) < t$. We can compute the WNF of the polynomial S-Poly(g_j, g_k) by G and we get some invertible polynomial $u_{j,l}$ and polynomials $v_i^{(j,l)}$ such that: $u_{j,l} \text{S-Poly}(g_j, g_l) = \sum_{i=1}^s v_i^{(j,l)} g_i$ with $\mathrm{LT}_{\mathbf{r}}(v_i^{(j,l)} g_i) \leq \mathrm{LT}_{\mathbf{r}}(\text{S-Poly}(g_j, g_l))$.

Multiplying by $u_{j,l}^{-1}$ and summing those decompositions, we get that $\sum_{j,l \in J} t_{j,l}' \text{S-Poly}(g_j, g_l) = \sum_{i=1}^s w_i g_i$ with $\mathrm{LT}_{\mathbf{r}}(w_i g_i) \leq \max_{j,l} \mathrm{LT}_{\mathbf{r}}(u_{j,l}^{-1} v_i^{(j,l)} g_i) = \max_{j,l} \mathrm{LT}_{\mathbf{r}}(v_i^{(j,l)} g_i)$. So $\mathrm{LT}_{\mathbf{r}}(w_i g_i)$ is less than or equal to $\max_{j,l} \mathrm{LT}_{\mathbf{r}}(t_{j,l}' \text{S-Poly}(g_j, g_l))$ and strictly smaller than t. Summing all summands we then obtain a new decomposition of f contradicting the minimality of t. □

Algorithm 3 Buchberger's algorithm with Mora's WNF

Input: $F := (f_1, \ldots, f_s)$ a list of polynomials in $K[\mathbf{X}]$.
Output: G a list of polynomials in $K[\mathbf{X}]$ which is a GB of $\langle F \rangle$

```
G := F ;
P := {(f, g) | f, g ∈ G, f ≠ g} ;
while P ≠ ∅ do
    choose and remove (f, g) from P ;
    h := WNF(S-Poly(f, g), G);
    if h ≠ 0 then
        P := P ∪ {(h, f) | f ∈ G} ;
        G := G ∪ {h} ;
return G ;
```

Proposition 5.3. *Algorithm 3 terminates and is correct.*

Proof. Correctness comes from Buchberger's criterion. Termination is a consequence of Prop 2.8 of [4]. □

Corollary 5.4. *If I is generated by polynomials, then Algorithm 3 provides a GB of $I_{\mathbf{r}} \subset K\{\mathbf{X}; \mathbf{r}\}$ made of polynomials of I.*

Proof. If $f \in I$ and $G \subset I$ are polynomials, then $\mathsf{WNF}(f, G)$ is a polynomial of I. As the S-Poly considered in Algo. 3 are polynomials in I, we obtain the result. □

5.2 Precision and effective computations

We may remark firstly that, as we wrote all properties and proofs in terms of LT's, the algorithms of §3 and 4 are valid over $K\{\mathbf{X}; \mathbf{r}\}^{\circ}$. In particular, if we work with $\mathbf{r} = (0, \ldots, 0)$, then no division in K is involved: as in [4], working at finite precision, no loss of absolute precision can occur.

Secondly, if we work in $K^{\mathrm{ex}} \subset K$, all computations take place in K^{ex}. Hence if $(f_1, \ldots, f_s) \in K^{\mathrm{ex}}[\mathbf{X}]$ and $\mathbf{r} \in \mathbb{Q}^n$, Algorithms 1 and 3 provide an algorithm working over K^{ex} to compute a GB of $I_{\mathbf{r}}$ made of polynomials in $K^{\mathrm{ex}}[\mathbf{X}]$, without having to deal with any precision issue.

5.3 Toy Implementation

A toy implementation of the algorithms of this Section is available here: https://gist.github.com/TristanVaccon. We present some timings and features of the Algorithm in Appendix 8 on page 9.

6 MORA'S WNF AND OVERCONVERGENCE

We now consider the case of overconvergent series, and present a version of Mora's weak normal form algorithm for that case.

6.1 Écarts for overconvergence

Let $f, g \in K\{\mathbf{X}; \mathbf{s}\}, \mathbf{s} \in \mathbb{Q}^n, \mathbf{r} \in \mathbb{Q}^n, \mathbf{s} \geq \mathbf{r}$. We define écarts adapted to computation over $K\{\mathbf{X}; \mathbf{r}\}$ for series belonging also to $K\{\mathbf{X}; \mathbf{s}\}$.

Definition 6.1. We define the $\mathbf{s}$-support of $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \mathbf{X}^\alpha \in K\{\mathbf{X}; \mathbf{s}\}$ as:

$$\mathrm{Supp}_{\mathbf{s}}(f) = \left\{ \alpha \text{ s.t. } \mathrm{val}_{\mathbf{s}}(c_\alpha \mathbf{X}^\alpha) = \mathrm{val}_{\mathbf{s}}(f) \right\}.$$

Since $f \in K\{\mathbf{X}; \mathbf{s}\}$, $\mathrm{Supp}_{\mathbf{s}}(f)$ is finite. Then, we define the $(\mathbf{s}, \mathbf{r})$-degree of f as:

$$\deg_{\mathbf{s},\mathbf{r}}(f) = \max_{\alpha \in \mathrm{Supp}_{\mathbf{s}}(f)} (\mathbf{s} - \mathbf{r}) \cdot \alpha.$$

Definition 6.2. We define:

$$\text{Écart}_{\mathbf{s},\mathbf{r},0}(f) := \mathrm{val}_{\mathbf{s}}(\mathrm{LT}_{\mathbf{r}}(f)) - \mathrm{val}_{\mathbf{s}}(f),$$
$$\text{Écart}_{\mathbf{s},\mathbf{r},1}(f) := \deg_{\mathbf{s},\mathbf{r}}(f) - \deg_{\mathbf{s},\mathbf{r}}(\mathrm{LT}_{\mathbf{r}}(f))$$

Lemma 6.3. *For $f \in K\{\mathbf{X}; \mathbf{s}\}$, $i \in \{0, 1\}$, $\text{Écart}_{\mathbf{s},\mathbf{r},i}(f) \geq 0$.*

Proof. For $\text{Écart}_{\mathbf{s},\mathbf{r},0}(f)$, it is a direct consequence of the definition of $\mathrm{val}_{\mathbf{s}}$.

Now, let us take some $\alpha \in \mathrm{Supp}_{\mathbf{s}}(f)$ such that $(\mathbf{s} - \mathbf{r}) \cdot \alpha = \deg_{\mathbf{s},\mathbf{r}}(f)$. Let c_α be the coefficient of $\mathbf{X}^\alpha$ in f. Let $\mathrm{LT}_{\mathbf{r}}(f) = c_\beta \mathbf{X}^\beta$. Then, by definition, $\mathrm{val}_{\mathbf{s}}(c_\beta \mathbf{X}^\beta) \geq \mathrm{val}_{\mathbf{s}}(f) = \mathrm{val}_{\mathbf{s}}(c_\alpha \mathbf{X}^\alpha)$, and $\mathrm{val}_{\mathbf{r}}(c_\alpha \mathbf{X}^\alpha) \geq \mathrm{val}_{\mathbf{r}}(c_\beta \mathbf{X}^\beta)$. Thus,

$$\mathrm{val}(c_\beta) - \mathbf{s} \cdot \beta \geq \mathrm{val}(c_\alpha) - \mathbf{s} \cdot \alpha,$$
$$\mathrm{val}(c_\alpha) - \mathbf{r} \cdot \alpha \geq \mathrm{val}(c_\beta) - \mathbf{r} \cdot \beta, \text{ and then}$$
$$\mathbf{s} \cdot (\alpha - \beta) \geq \mathrm{val}(c_\alpha) - \mathrm{val}(c_\beta) \geq \mathbf{r} \cdot (\alpha - \beta),$$

which implies that $(\mathbf{s} - \mathbf{r}) \cdot (\alpha - \beta) \geq 0$. Since $\deg_{\mathbf{s},\mathbf{r}}(f) = (\mathbf{s} - \mathbf{r}) \cdot \alpha$ and $\deg_{\mathbf{s},\mathbf{r}}(\mathrm{LT}_{\mathbf{r}}(f)) = (\mathbf{s} - \mathbf{r}) \cdot \beta$, we can conclude that $\text{Écart}_{\mathbf{s},\mathbf{r},1}(f) \geq 0$. □

6.2 WNF algorithm for overconvergent series

The algorithm is straightforward, using the adapted notions of écarts.

Algorithm 4 $\mathsf{WNF}(f, g, s, r)$, Mora's overconvergent Weak Normal Form algorithm

Input: $f, g_1, \ldots, g_s \in K\{\mathbf{X}; \mathbf{s}\}, \mathbf{s} \in \mathbb{Q}^n, \mathbf{r} \in \mathbb{Q}^n, \mathbf{s} \geq \mathbf{r}$
Output: $h \in K\{\mathbf{X}; \mathbf{s}\}$ such that for some $\mu, u_1, \ldots, u_s \in K\{\mathbf{X}; \mathbf{s}\}$, $\mu f = \sum u_i g_i + h$,
when $h \neq 0$, $\mathrm{LT}_{\mathbf{r}}(h)$ is divisible by no $\mathrm{LT}_{\mathbf{r}}(g_i)$'s and μ is invertible in $K\{\mathbf{X}; \mathbf{r}\}$. Moreover, $\mathrm{LT}_{\mathbf{r}}(u_i g_i) \leq \mathrm{LT}_{\mathbf{r}}(f)$.

```
h := f ;
T := (g_1, ..., g_s) ;
while h ≠ 0 and T_h := {g ∈ T, LT_r(g) | LT_r(h)} ≠ ∅ do
    choose g ∈ T_h minimizing first Écart_{s,r,0}(g), then Écart_{s,r,1}(g) ;
    if Écart_{s,r,0}(g) > Écart_{s,r,0}(h), or Écart_{s,r,1}(g) > Écart_{s,r,1}(h) then
        T := T ∪ {h};
    h := S-Poly(h, g) ;
return h ;
```

6.3 Correctness and convergence

Lemma 6.4. *If $g \in T_{h_m}$ is such that:*

- $\text{Écart}_{\mathbf{s},\mathbf{r},0}(g) \leq \text{Écart}_{\mathbf{s},\mathbf{r},0}(h_m)$,
- $\text{Écart}_{\mathbf{s},\mathbf{r},1}(g) \leq \text{Écart}_{\mathbf{s},\mathbf{r},1}(h_m)$,

and if $t = \mathrm{LT}_{\mathbf{r}}(h_m)/\mathrm{LT}_{\mathbf{r}}(g)$ and $h_{m+1} = h_m - tg$, then

$$\mathrm{val}_s(h_{m+1}) \geq \mathrm{val}_s(h_m).$$

In case of equality, then moreover,

$$\deg_{\mathbf{s},\mathbf{r}}(h_{m+1}) \leq \deg_{\mathbf{s},\mathbf{r}}(h_m).$$

Proof. Since $\text{Écart}_{\mathbf{s},\mathbf{r},0}(g) \leq \text{Écart}_{\mathbf{s},\mathbf{r},0}(h_m)$ and $\text{Écart}_{\mathbf{s},\mathbf{r},0}(g) = \text{Écart}_{\mathbf{s},\mathbf{r},0}(tg)$, then $\text{val}_{\mathbf{s}}(\text{LT}_{\mathbf{r}}(tg)) - \text{val}_{\mathbf{s}}(tg) \leq \text{val}_{\mathbf{s}}(\text{LT}_{\mathbf{r}}(h_m)) - \text{val}_{\mathbf{s}}(h_m)$. Moreover, $\text{LT}_{\mathbf{r}}(tg) = \text{LT}_{\mathbf{r}}(h_m)$, so $\text{val}_{\mathbf{s}}(tg) \geq \text{val}_{\mathbf{s}}(h_m)$. By the ultrametric inequality, we then obtain that $\text{val}_{\mathbf{s}}(h_{m+1}) \geq \text{val}_{\mathbf{s}}(h_m)$.

Now, if $\text{val}_{\mathbf{s}}(h_{m+1}) \geq \text{val}_{\mathbf{s}}(h_m)$, we prove that $\deg_{\mathbf{s},\mathbf{r}}(h_{m+1}) \leq \deg_{\mathbf{s},\mathbf{r}}(h_m)$. Since $\text{Écart}_{\mathbf{s},\mathbf{r},1}(g) = \text{Écart}_{\mathbf{s},\mathbf{r},1}(tg)$, then the second hypothesis means that $\deg_{\mathbf{s},\mathbf{r}}(tg) - \deg_{\mathbf{s},\mathbf{r}}(\text{LT}_{\mathbf{r}}(tg)) \leq \deg_{\mathbf{s},\mathbf{r}}(h_m) - \deg_{\mathbf{s},\mathbf{r}}(\text{LT}_{\mathbf{r}}(h_m))$. From the equality $\text{LT}_{\mathbf{r}}(tg) = \text{LT}_{\mathbf{r}}(h_m)$, it follows that $\deg_{\mathbf{s},\mathbf{r}}(tg) \leq \deg_{\mathbf{s},\mathbf{r}}(h_m)$. As $h_{m+1} = h_m - tg$, then $\deg_{\mathbf{s},\mathbf{r}}(h_{m+1}) \leq \max\left(\deg_{\mathbf{s},\mathbf{r}}(h_m), \deg_{\mathbf{s},\mathbf{r}}(tg)\right)$, and we can conclude. ◻

Proposition 6.5. *If $\mathbf{r} < \mathbf{s}$ then either Algorithm 4 terminates in a finite number of steps, or both $\text{LT}_{\mathbf{r}}(h_m)$ and $\text{LT}_{\mathbf{s}}(h_m)$ converge to 0.*

Proof. Let us assume that Algorithm 4 does not terminate for some inputs $\mathbf{s} > \mathbf{r}$ and $f, g_1, \dots, g_s \in K\{\mathbf{X}; \mathbf{s}\}$.

As we do eliminate successively the $\text{LT}_{\mathbf{r}}(h_m)$'s, then by design, $\text{LT}_{\mathbf{r}}(h_m)$ converges to zero.

Let $d_1, d_2 \in \mathbb{N}$ be such that for any $f \in K\{\mathbf{X}; \mathbf{s}\}$, $d_1 \text{val}_{\mathbf{s}}(f) \in \mathbb{Z}$ and $d_2 \deg_{\mathbf{s},\mathbf{r}}(f) \in \mathbb{Z}$.

Let us define the extended leading term of $h \in K\{\mathbf{X}; \mathbf{s}\}$ as: $\text{LTE}(h) := U^{d_1 \text{Écart}_{\mathbf{s},\mathbf{r},0}(h)} V^{d_2 \text{Écart}_{\mathbf{s},\mathbf{r},1}(h)} \text{LT}(h) \in K[\mathbf{X}, U, V]$.

Then, there is some $N_1 \in \mathbb{N}$ such that for $m \geq N_1$, the monomial ideal of $K[\mathbf{X}, U, V]$ generated by the LTE's of the series in T is constant (thanks to Prop. 2.8 of [4]). Thus for $m \geq N_1$, if h_m is not added to T at the end of the **while** loop, then there is some $g \in T_{h_m}$ such that $\text{Écart}_{\mathbf{s},\mathbf{r},0}(g) \leq \text{Écart}_{\mathbf{s},\mathbf{r},0}(h_m)$. If it is added, then by definition of N_1, it means that there is some $g \in T$ such that $\text{LTE}(g)$ divides $\text{LTE}(h_m)$, and this implies that $\text{LT}(g) \mid \text{LT}(h_m)$ and $\text{Écart}_{\mathbf{s},\mathbf{r},0}(g) \leq \text{Écart}_{\mathbf{s},\mathbf{r},0}(h_m)$.

So in both cases, $\text{Écart}_{\mathbf{s},\mathbf{r},0}(g) \leq \text{Écart}_{\mathbf{s},\mathbf{r},0}(h_m)$.

Then, if h_m is not added to T, it means that the minimal g satisfies $\text{Écart}_{\mathbf{s},\mathbf{r},1}(g) \leq \text{Écart}_{\mathbf{s},\mathbf{r},1}(h_m)$. If it is added to T, then again, by definition of N_1, it means that there is some $g \in T$ such that $\text{LTE}(g)$ divides $\text{LTE}(h_m)$, and this implies that $\text{LT}(g) \mid \text{LT}(h_m)$ and $\text{Écart}_{\mathbf{s},\mathbf{r},0}(g) \leq \text{Écart}_{\mathbf{s},\mathbf{r},0}(h_m)$ and $\text{Écart}_{\mathbf{s},\mathbf{r},1}(g) \leq \text{Écart}_{\mathbf{s},\mathbf{r},1}(h_m)$. So in both cases, the minimal g for the reduction satisfies that $\text{Écart}_{\mathbf{s},\mathbf{r},1}(g) \leq \text{Écart}_{\mathbf{s},\mathbf{r},1}(h_m)$.

We can then apply Lemma 6.4: for any $m \geq N_1$, $\text{val}_{\mathbf{s}}(h_{m+1}) \geq \text{val}_{\mathbf{s}}(h_m)$ and in case of equality, $\deg_{\mathbf{s},\mathbf{r}}(h_{m+1}) \leq \deg_{\mathbf{s},\mathbf{r}}(h_m)$.

Consequently, $\text{val}_{\mathbf{s}}(h_m)$ is a non-decreasing sequence in $\frac{1}{d_1}\mathbb{Z}$. Hence, either it goes to $+\infty$, or there is some $N_2 \geq N_1$ such that $\text{val}_{\mathbf{s}}(h_m)$ is constant for $m \geq N_2$.

Let us assume that we are in this second case. Then $\deg_{\mathbf{s},\mathbf{r}}(h_m)$ is non-increasing (for $m \geq N_2$) and thus, upper-bounded. Let $m \geq N_2$ and $t = c_\alpha \mathbf{X}^\alpha$ a term of h_m in $\text{Supp}_{\mathbf{s}}(h_m)$.

Then $\text{val}_{\mathbf{s}}(h_m) = \text{val}_{\mathbf{s}}(t)$ and

$$\begin{aligned} \text{val}_{\mathbf{r}}(h_m) &\leq \text{val}_{\mathbf{r}}(t) \leq \text{val}_{\mathbf{s}}(t) + (\mathbf{s} - \mathbf{r}) \cdot \alpha \\ &\leq \text{val}_{\mathbf{s}}(h_m) + \deg_{\mathbf{s},\mathbf{r}}(h_m). \end{aligned}$$

Both $\text{val}_{\mathbf{s}}(h_m)$ and $\deg_{\mathbf{s},\mathbf{r}}(h_m)$ are upper-bounded, while $\text{val}_{\mathbf{r}}(h_m) \to +\infty$. This is a contradiction.

Consequently, $\text{val}_{\mathbf{s}}(h_m) \to +\infty$, which concludes the proof. ◻

Proposition 6.6. *Algorithm 4 is correct and mutatis mutandis, computes a weak normal form.*

Proof. *Mutatis mutandis*, the loop invariant in Lemma 4.5 is still valid. When f does not reduce to zero by $g_1, \dots, g_s$, there is no difficulty as Algorithm 4 terminates in a finite number of steps, and $\mu, g_1, \dots, g_s$ are polynomials, with μ invertible in $K\{\mathbf{X}; \mathbf{r}\}$. When f reduces to zero, we proved in Lemma 6.4, that $\text{val}_s(h_m)$ is eventually increasing and going to $+\infty$. We showed in the proof of Prop. 6.5 that eventually, T is constant. It then proves that, for the g on Line 4, for $c_\nu x^\nu = \left(\frac{\text{LT}_{\mathbf{r}}(h_m)}{\text{LT}_{\mathbf{r}}(g)}\right)$, then $\text{val}_{\mathbf{s}}(c_\nu x^\nu) \to +\infty$. This is enough to prove that the $\mu, u_1, \dots, u_s$ such that $\mu f = \sum_i u_i g_i$ are in $K\{\mathbf{X}; \mathbf{s}\}$ as expected. ◻

Remark 6.7. Section 5 can be extended with (almost) no modification to compute GB in $K\{\mathbf{X}; \mathbf{s}\}$ of an $\mathbf{s}$-convergent ideal of $K\{\mathbf{X}; \mathbf{r}\}$. One just needs to replace $K[\mathbf{X}]$ by $K\{\mathbf{X}; \mathbf{s}\}$ and use Algo. 4 in Buchberger's algorithm.

7 UNIVERSAL GRÖBNER BASIS

In this Section, we prove that a polynomial ideal can only have a finite number of distinct initial ideals for varying log-radii $\mathbf{r}$. To do so, we first prove the result for homogeneous ideals by adapting the classical proof for polynomial ideals and then use homogenization to generalize the result to non-homogeneous ideals.

7.1 Homogeneous ideal

The classical proof that a polynomial ideal has only finitely many initial ideals from page 427 of [10] (see also [19]) can be adapted to our setting by relying on the following Lemma.

Lemma 7.1. *If $I \subset K[\mathbf{X}]$ is a homogeneous ideal, if $\mathbf{r} \in \mathbb{Q}^n$, and if $F = (f_1, \dots, f_s) \in I^s$ are homogeneous polynomials which do not form a GB of $I_{\mathbf{r}}$, then there exists some homogeneous polynomial $g \in I$ such that no term of g is divisible by any of the $\text{LT}_{\mathbf{r}}(f_i)$'s.*

Proof. Since F is not a GB of $I_{\mathbf{r}}$, there exists some term $cx^\alpha \in \text{LT}_{\mathbf{r}}(I_{\mathbf{r}})$ such that $cx^\alpha \notin \langle \text{LT}_{\mathbf{r}}(f_1), \dots, \text{LT}_{\mathbf{r}}(f_s) \rangle$. By the density of I in $I_{\mathbf{r}}$ there is some polynomial $h \in I$ such that $\text{LT}_{\mathbf{r}}(h) = cx^\alpha$. Since I is homogeneous, we can assume that so is h.

By performing the tropical row-echelon algorithm of [21] (Algorithm 1) on a Macaulay matrix consisting of h and the multiples of the elements of F of degree $\deg(h)$, we obtain g such that no term of g is divisible by any of the $\text{LT}_{\mathbf{r}}(f_i)$'s. ◻

Using linear algebra along the same lines, we get the existence of polynomial reduced Gröbner bases.

Lemma 7.2. *If $I \subset K[\mathbf{X}]$ is a homogeneous ideal, if $\mathbf{r} \in \mathbb{Q}^n$, then there exists G a reduced Gröbner basis of $I_{\mathbf{r}}$ made of finitely many homogeneous polynomials of I.*

Proof. Thanks to Corollary 5.4, we get H, a GB or $I_{\mathbf{r}}$ made of polynomials of I. Since I is homogeneous we can assume that in addition, they are all homogeneous. Then again, for any $g \in G$, we can perform inter-reduction by performing the tropical row-echelon algorithm of [21] (Algorithm 1) on a Macaulay matrix consisting of g and the multiples of the elements of $G \setminus \{g\}$ of degree $\deg(g)$. This is enough to conclude. ◻

Proposition 7.3. *Let $I \subset K[\mathbf{X}]$ be a homogeneous ideal. Then the set $Terms(I) := \{\mathrm{LT}(I_{\mathbf{r}}) \ / \ \mathbf{r} \in \mathbb{Q}^n\}$ is finite.*

Proof. Suppose that $\mathrm{Terms}(I)$ is infinite. For any $M \in \mathrm{Terms}(I)$, we write $\leq_M$ for a term order defined by an $\mathbf{r}$ such that $\mathrm{LT}(I_{\mathbf{r}}) = M$. Let $\Sigma := \{\leq_M \ / \ M \in \mathrm{Terms}(I)\}$. Our assumption states that Σ is infinite.

Let $f_1 \in I$ be a homogeneous polynomial. Since f_1 has finitely many terms, by the pigeonhole principle, there is an infinite set $\Sigma_1 \subset \Sigma$ and a term m_1 of f_1 such that for all $\leq_M \in \Sigma_1$, $\mathrm{LT}_{\leq_M}(f_1) = m_1$. Suppose that for some $\leq_1 \in \Sigma_1$ defined by some $\mathbf{r}_1$, (f_1) is a GB of $I_{\mathbf{r}_1}$. Then, let $\leq \in \Sigma_1$ be defined by some $\mathbf{r}$. We prove that (f_1) is then a GB of $I_{\mathbf{r}}$. Indeed, by Lemma 7.1, if (f_1) is not a GB of $I_{\mathbf{r}}$ there is some $h \in I$ such that no term of h is divisible by $\mathrm{LT}_{\leq}(f_1)$. Since $\mathrm{LT}_{\leq}(f_1) = \mathrm{LT}_{\leq_1}(f_1)$ and (f_1) is a GB of $I_{\mathbf{r}_1}$ this is a contradiction. Consequently, for any $\leq \in \Sigma_1$ defined by some $\mathbf{r}$, (f_1) is a GB of $I_{\mathbf{r}}$ with $\mathrm{LT}_{\leq}(f_1) = m_1$. However, this can not be the case as our assumption was that there are infinitely many elements in Σ_1 all defining distinct LT's for I. Therefore, (f_1) is not a GB of $I_{\mathbf{r}}$.

By Lemma 7.1 there is some homogeneous $f_2 \in I$ such that no term of f_2 is divisible by m_1. Then again, since f_2 has finitely many terms, by the pigeonhole principle, there is an infinite set $\Sigma_2 \subset \Sigma_1$ and a term m_2 of f_2 such that for all $\leq_M \in \Sigma_2$, $\mathrm{LT}_{\leq_M}(f_2) = m_2$ (and also since $\Sigma_2 \subset \Sigma_1$, $\mathrm{LT}_{\leq_M}(f_1) = m_1$).

The same argument as above shows that for any $\leq \in \Sigma_2$ defined by some $\mathbf{r}$, (f_1, f_2) is not GB of $I_{\mathbf{r}}$. Then again, by Lemma 7.1 there is some homogeneous $f_3 \in I$ such that no term of f_3 is divisible by any of (m_1, m_2). Since f_3 has finitely many terms, by the pigeonhole principle, there is an infinite set $\Sigma_3 \subset \Sigma_2$ and a term m_3 of f_3 such that for all $\leq_M \in \Sigma_3$, $\mathrm{LT}_{\leq_M}(f_3) = m_3$ (and also since $\Sigma_3 \subset \Sigma_2$, $\mathrm{LT}_{\leq_M}(f_1) = m_1, \mathrm{LT}_{\leq_M}(f_2) = m_2$).

Continuing the same way, we produce a descending chain of infinite subsets $\Sigma \supset \Sigma_1 \supset \Sigma_2 \supset \Sigma_3 \supset \dots$ and an infinite strictly ascending chain of ideals $\langle m_1 \rangle \subset \langle m_1, m_2 \rangle \subset \langle m_1, m_2, m_3 \rangle \subset \dots$ in $\mathbb{T}\{\mathbf{X}\}$. This contradicts Prop. 2.8 of [4] and concludes the proof. ◻

Theorem 7.4. *Let $I \subset K[\mathbf{X}]$ be a homogeneous ideal. Then there exists a finite set $G \subset I \subset K[\mathbf{X}]$ made of homogeneous polynomials which is a universal analytic Gröbner basis of I: for any $\mathbf{r} \in \mathbb{Q}^n$, G is a GB of I_r.*

Proof. By Prop 7.3, there are only finitely many initial ideals possible. We prove that for two term-orders $\leq_1$ and $\leq_2$ (defined by $\mathbf{r}_1$ and $\mathbf{r}_2$), if they define the same initial ideal, then they have the same reduced Gröbner basis. Indeed, let G_1 and G_2 be the reduced Gröbner bases given by Lemma 7.2. They have the same LT's. Let $g_1 \in G_1$ and $g_2 \in G_2$ having a common LT. Then $g_1 - g_2 \in I$ with no monomial divisible by any of the $\mathrm{LT}(G_i)$'s. Hence $g_1 = g_2$, and G_1 and G_2 are equal up to permutation.

Consequently, all term orders giving rise to the same initial ideal share the same reduced GB. Consequently, only a finite amount of reduced GB for the $I_{\mathbf{r}}$'s are possible. By concatening all of them, we obtain the desired universal analytic Gröbner basis. ◻

7.2 Non-Homogeneous ideal

Lemma 7.5. *Let $I \subset K[\mathbf{X}]$ be a polynomial ideal and $\mathbf{r} \in \mathbb{Q}^n$. Let $(h_1, \dots, h_s)$ be a finite Gröbner basis of $(I^*)_{(r,0)} \subset K\{\mathbf{X}, t; \mathbf{r}, 0\}$ made of homogeneous polynomials of I^* (hence in $K[\mathbf{X}, t]$). Then $(h_{1,*}, \dots, h_{s,*})$ is a Gröbner basis of I_r.*

Proof. Firstly, due to being dehomogenization of elements of I^*, the $h_{i,*}$'s are in I.

Secondly, by Corollary 5.4, it is enough to check that for any $f \in I$, $\mathrm{LT}_{\mathbf{r}}(f)$ is divisible by one of the $\mathrm{LT}_{\mathbf{r}}(h_{i,*})$'s.

Let $f \in I$. Then $f^* \in I^* \subset (I^*)_{(r,0)}$ so there is some i such that $\mathrm{LT}_{(\mathbf{r},0)}(h_i)$ divides $\mathrm{LT}_{(\mathbf{r},0)}(f^*)$. Then thanks to Lemma 3.5, $\mathrm{LT}_{\mathbf{r}}(f) = \mathrm{LT}_{(\mathbf{r},0)}(f^*)_*$, $\mathrm{LT}_{\mathbf{r}}(h_{i,*}) = \mathrm{LT}_{(\mathbf{r},0)}(h_i)_*$, and monomial divisibility is preserved by dehomogenization. So $\mathrm{LT}_{\mathbf{r}}(h_{i,*})$ divides $\mathrm{LT}_{\mathbf{r}}(f)$ and the proof is complete. ◻

We can then prove the main theorem of this section.

Theorem 7.6. *Let $I \subset K[\mathbf{X}]$ be an ideal. Then the set $Terms(I) := \{\mathrm{LT}(I_{\mathbf{r}}) \ / \ \mathbf{r} \in \mathbb{Q}^n\}$ is finite.*

Proof. Thanks to Lemma 7.5, there is a surjection from $\mathrm{Terms}(I^*)$ to $\mathrm{Terms}(I)$. The first set is finite thanks to Proposition 7.3, so the second is also, which concludes the proof. ◻

We can also obtain the existence of universal Gröbner bases for any polynomial ideal in $K[\mathbf{X}]$.

Theorem 7.7. *Let $I \subset K[\mathbf{X}]$ be an ideal. Then there exists a finite set $G \subset I \subset K[\mathbf{X}]$ which is a universal analytic Gröbner basis of I: for any $\mathbf{r} \in \mathbb{Q}^n$, G is a GB of I_r.*

Proof. Thanks to Lemma 7.5, it is enough to dehomogenize a universal analytic GB of I^* to obtain the desired universal analytic GB of I. ◻

7.3 New challenges

One can relate the previous result to the Remark 8.8 of [17] on the foundations of computations in tropical analytic geometry, on universal analytic GB and on tropical bases.

We say that $F \subset I$ is a tropical basis of I if for any $\mathbf{r} \in \mathbb{Q}^n$: there is $g \in I$ such that $\mathrm{val}_{\mathbf{r}}(g)$ is reached by only one term if and only if there is $f \in F$ such that $\mathrm{val}_{\mathbf{r}}(f)$ is reached by only one term.

It leaves us with the following challenges:

(1) Give an algorithm to compute a universal analytic Gröbner basis of a polynomial ideal.
(2) Give an algorithm to compute a tropical basis of a polynomial ideal.
(3) Generalize universal analytic GB to overconvergent ideals or to varying center of polydisks of convergence.

We shall remark that in our context, due to the fact that we take the valuation of the coefficients into account, then, contrary to the classical case of Gröbner fans for polynomials over a field, the Gröbner complex is in general not a cone.

Timings (s)			Entry precision in $\mathbb{Q}_p[\mathbf{X}]$ or $\mathbb{Q}_p\{\mathbf{X};(0,\dots,0)\}$						
system	p	**algo**	2^4	2^5	2^6	2^7	2^8	2^9	2^{20}
Cyclic 5	2	Mora	∞	∞	∞	∞	∞	∞	∞
		Vapote	0.86	1.0	1.5	2.3	3.8	7.2	∞
Katsura 3	2	Mora	0.031	0.047	0.031	0.063	0.047	0.032	0.5
		Vapote	0.063	2.2	140	4500	∞	∞	∞
Katsura 6	2	Mora	1.2	0.98	0.94	1.0	1.1	1.0	2.3
		Vapote	170	∞	∞	∞	∞	∞	∞

Table 1: Precision and timing for Algo 3 and Vapote [5]. The log-radii is $(0,\dots,0)$.

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8 APPENDIX: TIMINGS

We present here with Table 1 some timings for our toy implementation of Algorithm 3 acting on special systems in $\mathbb{Q}[\mathbf{X}] \subset \mathbb{Q}_2[\mathbf{X}] \subset \mathbb{Q}_2\{\mathbf{X};0,\dots,0\}$. The ∞ symbols means that 12 hours were not enough for the algorithm to terminate.

Most of the time the algorithms of [4, 5] vastly outperforms our implementation (as seen in the Cyclic case).

However, this is not always the case and with the Katsura systems, our implementation displays two remarkable features of our algorithm:

- Reductions can be significantly faster: no problem with reductions converging possibly slowly to zero [2]
- The dependency on the precision can be significantly smaller than that of the algorithms of [4, 5], allowing in some cases many orders of magnitude of additional digits in less time.

Please note the special shape of the Katsura 6 system in $\mathbb{Q}_p[X_1,\dots,X_6]$ for $\mathbf{r}=(0,\dots,0)$ and $p=2$: its defining polynomials already contains the leading monomials X_1, X_2, X_4, explaining in part why this computation is not as hard as for classical Gröbner bases.

One can try all examples at https://gist.github.com/TristanVaccon.

[2] An example of such a reduction slowly converging to 0 in the algorithms of [4, 5] is the reduction of X by $X - pX^2$ for log-radii 0, leading to intermediate remainders $pX^2, p^2X^3, \dots, p^kX^{k+1}, \dots$.