Computing Gröbner bases for structured systems

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Seminar *Algebra and discrete mathematics*, 24 November 2022

Recall from last semester

Complexity of computing Gröbner bases:

- In full generality (worst case): doubly exponential (≃ impossible)
- Generic: tractable with controlled complexity

Algorithm (Lazard 1983)

In. $F = \{f_1, \dots, f_m\}$ homogeneous polynomials (resp. degree d_i) in n variables, $D \in \mathbb{N}$

Out. G Gröbner basis up to degree D

1. For d from 0 to D
   1.1 Form the Macaulay matrix M of degree d of the system
   (matrix whose rows are all the mf_i with $\deg(m) + \deg(f_i) = d$)
   1.2 Echelon-reduce the matrix M
   1.3 Add to G each polynomial corresponding to a reduced row
2. Return G

Complexity bounds

Complexity of the algorithm = $O(DN_D^{\omega})$

- N_d: size of the matrix at degree d, $N_d = \binom{n+d-1}{d}$
- ω: exponent of the cost of reducing the matrices (in practice quite low thanks to sparsity)

What about D?

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Theorem (Hilbert): every ideal is finitely generated

Corollary (Buchberger): every ideal has a finite Gröbner basis

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Degree of regularity of the system: d_{reg} = smallest D such that the output is a Gröbner basis

Regular sequence: for all i, $qf_i \in \langle f_1, \dots, f_{i-1} \rangle \implies q \in \langle f_1, \dots, f_{i-1} \rangle$

"All reductions to 0 are predictable"

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"All reductions to 0 are predictable"

For homogeneous regular sequences with $m = n$: $d_{\mathrm{reg}} \leq \sum_{i=1}^{n} d_i - n + 1$

(generically sharp)

3 HYPOTHESES

The polynomial system must be:

- a regular sequence;
- square ($m = n$);
- homogeneous.

Can we relax those hypotheses?

Regularity: perhaps, but then there is nothing to keep us away from the worst case.

Under-determined systems $m < n$

Let $f_1, \ldots, f_m$ be a regular sequence and $I = \langle f_1, \ldots, f_m \rangle$.

The ideal I is **in Noether position** wrt $X_{m+1}, \ldots, X_n$

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… iff the canonical map $K[X_{m+1}, \dots, X_n] \to K[X_1, \dots, X_n]/I$ is injective
and such that $K[X_1, \dots, X_n]/I$ is integral over $K[X_{m+1}, \dots, X_n]$

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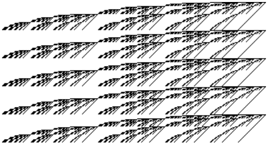
… iff $f_1, \dots, f_m, X_{m+1}, \dots, X_n$ is a regular sequence.

For homogeneous systems in Noether position: $d_{\mathrm{reg}} \leq \sum_{i=1}^{m} d_i - m + 1$ (generically sharp)

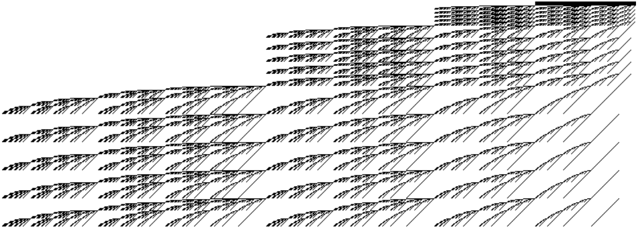
What about affine (non-homogeneous) systems? (1)



Macaulay matrix at deg 7 of a generic homogeneous system (5 variables, 5 polys, degree 4)

What about affine (non-homogeneous) systems? (1)

Adaptation of the algorithm: homogenize the system, or consider the lower degree monomials



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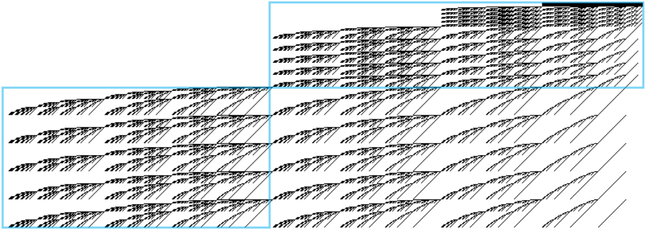


Macaulay matrix up to deg 7 of a generic affine system (5 variables, 5 polys, degree 4)

What about affine (non-homogeneous) systems? (2)

Size of the matrices: equivalent to adding an extra variable to the ring

Degree of regularity:

- if the highest degree components form a regular sequence,
  all degree falls are predictable,
  and we get the same bound as in the homogeneous case
- such a system is called a **regular sequence in the affine sense**
- if not, we don't have any bound

Sanity check: assume that $n = m$, let $f_1^h, \dots, f_n^h$ be the respective homogenizations (with the homogenization variable H):

$$f_1, \dots, f_n \text{ is a regular sequence in the affine sense}$$
$$\iff$$
$$f_1^h, \dots, f_n^h \text{ is in Noether position wrt } H$$

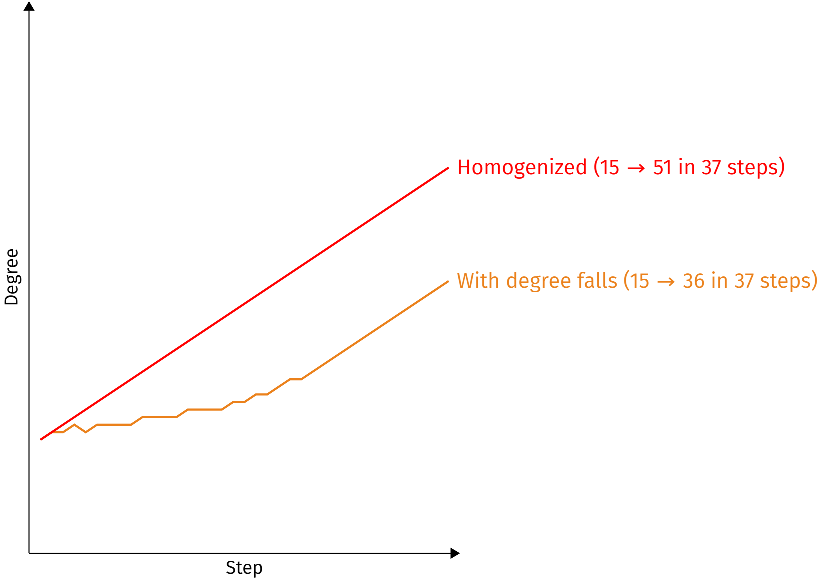
Problem

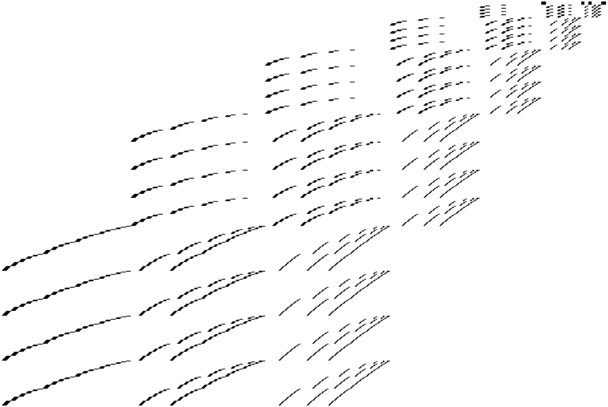
- There are other structures than homogeneity
- Systems with those structures are usually not regular in the affine sense
- They are also not instances of the worst-case complexity

Example

$$\begin{cases} x_1^5 + 43x_1^4x_2 + 96x_1^3x_2^2 + 115x_1^2x_2^3 + 118x_1x_2^4 + 91x_2^5 + 2x_1^3x_3 + 23x_1^2x_2x_3 + 60x_1x_2^2x_3 + 8x_2^3x_3 + 18x_1^3x_4 \\ \quad + 58x_1^2x_2x_4 + 125x_1x_2^2x_4 + 53x_2^3x_4 + 11x_1x_3^2 + 108x_2x_3^2 + 29x_1x_3x_4 + 9x_2x_3x_4 + 68x_1x_4^2 + 72x_2x_4^2 = 0 \\ x_1^5 + 23x_1^4x_2 + 34x_1^3x_2^2 + 21x_1^2x_2^3 + 22x_1x_2^4 + 70x_2^5 + 3x_1^3x_3 + 59x_1^2x_2x_3 + 17x_1x_2^2x_3 + 83x_2^3x_3 + 11x_1^3x_4 \\ \quad + 101x_1^2x_2x_4 + 61x_1x_2^2x_4 + 9x_2^3x_4 + 119x_1x_3^2 + 23x_2x_3^2 + 21x_1x_3x_4 + 69x_2x_3x_4 + 76x_1x_4^2 + 62x_2x_4^2 = 0 \\ x_1^5 + 21x_1^4x_2 + 2x_1^3x_2^2 + 81x_1^2x_2^3 + 98x_1x_2^4 + 61x_2^5 + 108x_1^3x_3 + 21x_1^2x_2x_3 + 37x_1x_2^2x_3 + 32x_2^3x_3 + 75x_1^3x_4 \\ \quad + 65x_1^2x_2x_4 + 49x_1x_2^2x_4 + 71x_2^3x_4 + 86x_1x_3^2 + 111x_2x_3^2 + 102x_1x_3x_4 + 78x_2x_3x_4 + 60x_1x_4^2 + 33x_2x_4^2 = 0 \\ x_1^5 + 77x_1^4x_2 + 117x_1^3x_2^2 + 56x_1^2x_2^3 + 89x_1x_2^4 + 36x_2^5 + 25x_1^3x_3 + 87x_1^2x_2x_3 + 90x_1x_2^2x_3 + 14x_2^3x_3 + 81x_1^3x_4 \\ \quad + 51x_1^2x_2x_4 + 24x_1x_2^2x_4 + 84x_2^3x_4 + 12x_1x_3^2 + 70x_2x_3^2 + 4x_1x_3x_4 + x_2x_3x_4 + 43x_1x_4^2 + 78x_2x_4^2 = 0 \end{cases}$$

Degree falls in the wild





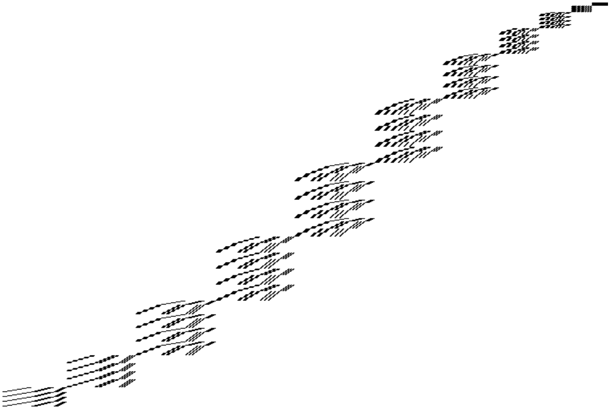
Affine Macaulay matrix up to deg 11 of the example system

Affine Macaulay matrix up to deg 11 of the example system
... after reordering the columns

Macaulay matrices for the example

Affine Macaulay matrix up to deg 11 of the example system
... after reordering the columns
... after reordering the rows

Macaulay matrices for the example



Affine Macaulay matrix up to deg 11 of the example system
… after reordering the columns
… after reordering the rows
… and after re-reordering the rows

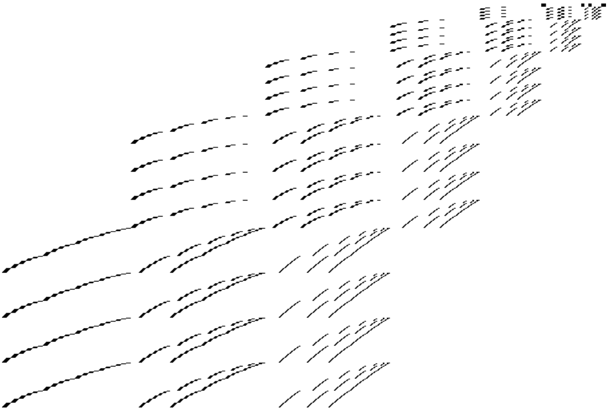
What was the reordering?

$$\begin{cases} x_1^5 + 43x_1^4x_2 + 96x_1^3x_2^2 + 115x_1^2x_2^3 + 118x_1x_2^4 + 91x_2^5 + 2x_1^3x_3 + 23x_1^2x_2x_3 + 60x_1x_2^2x_3 + 8x_2^3x_3 + 18x_1^3x_4 \\ \quad + 58x_1^2x_2x_4 + 125x_1x_2^2x_4 + 53x_2^3x_4 + 11x_1x_3^2 + 108x_2x_3^2 + 29x_1x_3x_4 + 9x_2x_3x_4 + 68x_1x_4^2 + 72x_2x_4^2 = 0 \\ (\ldots) \end{cases}$$

Weighted degree: $W = (w_1, \ldots, w_n) \in \mathbb{Z}^n$, $\deg_W\left(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}\right) = w_1\alpha_1 + \cdots + w_n\alpha_n$

The system is homogeneous for this weighted degree (**weighted-homogeneous**) for $W = (1, 1, 2, 2)$.

Macaulay matrices for the example



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… after reordering the columns by reverse W-degree

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Affine Macaulay matrix up to deg 11 of the example system
... after reordering the columns by reverse W-degree
... after reordering the rows by W-degree of the signatures

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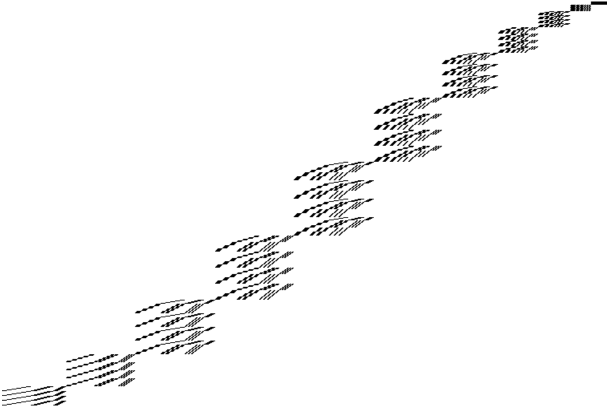

Affine Macaulay matrix up to deg 11 of the example system
... after reordering the columns by reverse W-degree
... after reordering the rows by W-degree of the signatures
... and after re-reordering the rows by W-degree of the polynomials

Are we happy?

No!

We have explained the structure of the Macaulay matrices with W-homogeneity,
but that doesn't tell us how to compute a Gröbner basis or estimate the complexity.

First idea: build the matrices weighted degree by weighted degree

Affine Macaulay matrix up to W-deg 11 of the example system

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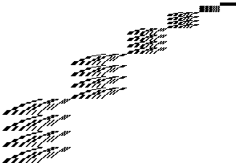
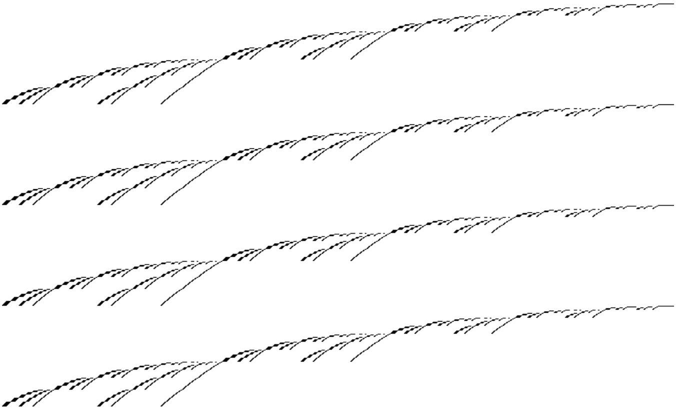


Affine Macaulay matrix up to W-deg 11 of the example system
… after reordering the rows and columns

Second idea: change of variable $x_i \mapsto x_i^{w_i}$



Affine Macaulay matrix up to deg 11 of the example system with $x_3 \mapsto x_3^2$ and $x_4 \mapsto x_4^2$

Weighted-homogeneous systems, strategy 2

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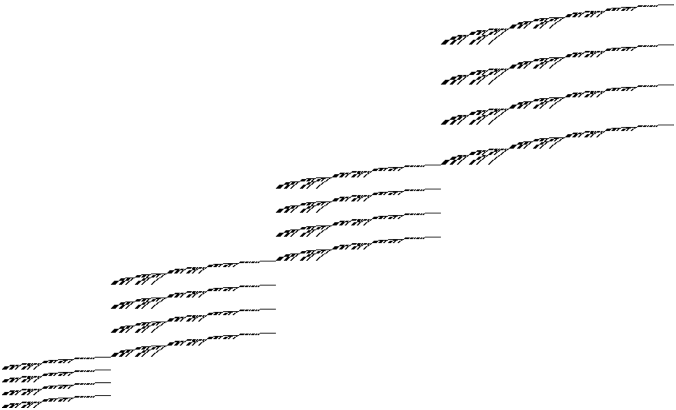

Affine Macaulay matrix up to deg 11 of the example system with $x_3 \mapsto x_3^2$ and $x_4 \mapsto x_4^2$
... after splitting according to the parity of the degrees in x_3 and x_4

The top-right corner is exactly the same as with the first strategy!

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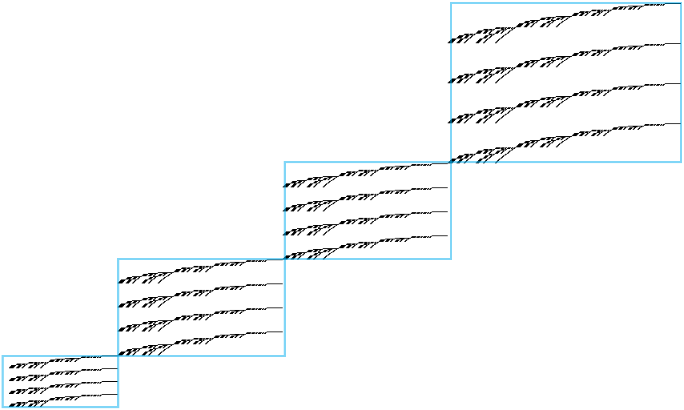

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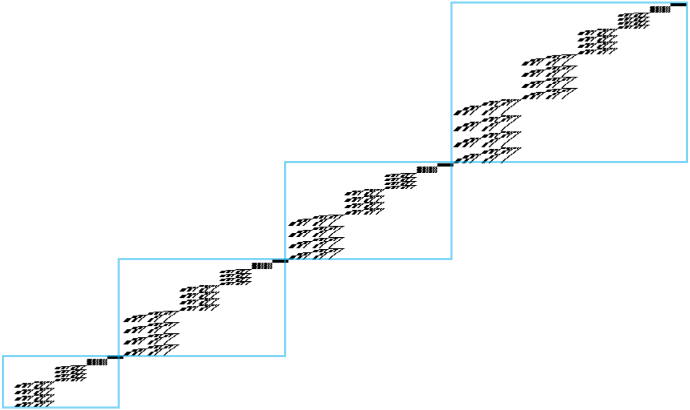

Affine Macaulay matrix up to deg 11 of the example system with $x_3 \mapsto x_3^2$ and $x_4 \mapsto x_4^2$
... after splitting according to the parity of the degrees in x_3 and x_4
... and after reordering

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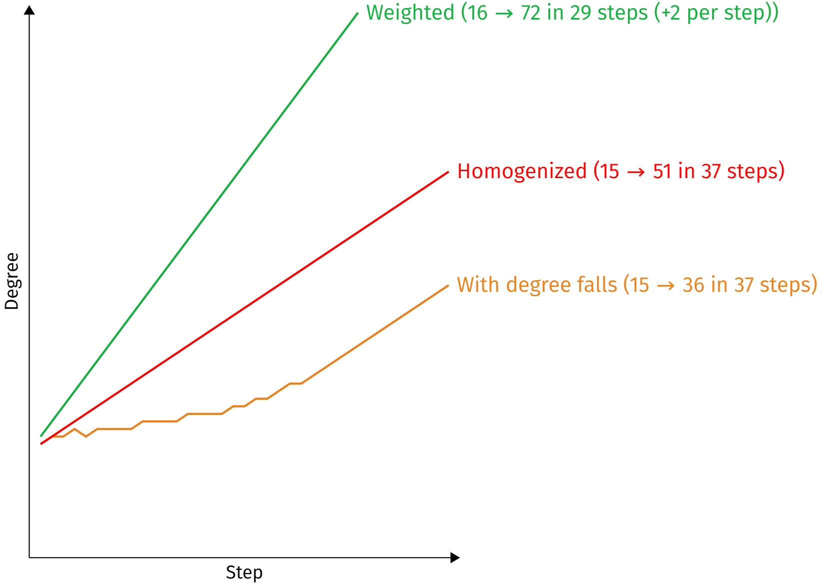
Degree falls in the wild



Size of the matrices: number of monomials at W-degree $d \simeq \frac{1}{\prod w_i}\binom{n+d-1}{d}$

Weighted degree of regularity: $(n = m)$

- for W-homo regular sequences: $d_{\mathrm{reg}} \leq \sum_{i=1}^{n}(d_i - w_i) + \max(w_i)$ (not generically sharp)
- for W-homo systems in simultaneous Noether position: $d_{\mathrm{reg}} \leq \sum_{i=1}^{n}(d_i - w_i) + w_n$ (generically sharp under some hypotheses on the weights)

Key ingredient: Hilbert series

Another example of structure

$$\begin{cases}
x_1y_1 + 23x_2y_1 + 18x_3y_1 + 54x_4y_1 + 78x_1y_2 + 71x_2y_2 + 32x_3y_2 + 24x_4y_2 + 64x_1y_3 \\
\quad + 90x_2y_3 + 78x_3y_3 + 65x_4y_3 + 94x_1y_4 + 72x_2y_4 + 114x_3y_4 + 30x_4y_4 = 0 \\
x_1y_1 + 62x_2y_1 + 88x_3y_1 + 117x_4y_1 + 68x_1y_2 + 79x_2y_2 + 125x_3y_2 + 106x_4y_2 + 47x_1y_3 \\
\quad + 125x_2y_3 + 13x_3y_3 + 92x_4y_3 + 43x_1y_4 + 119x_2y_4 + 23x_3y_4 + 39x_4y_4 = 0 \\
x_1y_1 + 112x_2y_1 + 39x_3y_1 + 39x_4y_1 + 105x_1y_2 + 12x_2y_2 + 60x_3y_2 + 61x_4y_2 + 77x_1y_3 \\
\quad + 40x_2y_3 + 28x_3y_3 + 120x_4y_3 + 24x_1y_4 + 115x_2y_4 + 121x_3y_4 + 24x_4y_4 = 0 \\
x_1y_1 + 5x_2y_1 + 28x_3y_1 + 56x_4y_1 + 41x_1y_2 + 14x_2y_2 + 89x_3y_2 + 96x_4y_2 + 40x_1y_3 \\
\quad + 69x_2y_3 + 114x_3y_3 + 31x_4y_3 + 117x_1y_4 + 75x_2y_4 + 41x_3y_4 + 46x_4y_4 = 0 \\
x_1y_1 + 13x_2y_1 + 4x_3y_1 + 117x_4y_1 + 57x_1y_2 + 112x_2y_2 + 4x_3y_2 + 9x_4y_2 + 90x_1y_3 \\
\quad + 121x_2y_3 + 5x_3y_3 + 74x_4y_3 + 97x_1y_4 + 67x_2y_4 + 83x_3y_4 + 82x_4y_4 = 0 \\
x_1y_1 + 92x_2y_1 + 34x_3y_1 + 24x_4y_1 + 95x_1y_2 + 118x_2y_2 + 76x_3y_2 + 75x_4y_2 + 69x_1y_3 \\
\quad + 34x_2y_3 + 51x_3y_3 + 4x_4y_3 + 2x_1y_4 + 35x_2y_4 + 121x_3y_4 + 43x_4y_4 = 0 \\
x_1y_1 + 72x_2y_1 + 95x_3y_1 + 117x_4y_1 + 24x_1y_2 + 72x_2y_2 + 39x_3y_2 + 75x_4y_2 + 93x_1y_3 \\
\quad + 57x_2y_3 + 58x_3y_3 + 83x_4y_3 + 41x_1y_4 + 50x_2y_4 + 90x_3y_4 + 20x_4y_4 = 0 \\
x_1y_1 + 88x_2y_1 + 72x_3y_1 + 108x_4y_1 + 65x_1y_2 + 120x_2y_2 + 123x_3y_2 + 15x_4y_2 + 123x_1y_3 \\
\quad + 81x_2y_3 + 77x_3y_3 + 121x_4y_3 + 95x_1y_4 + 47x_2y_4 + 28x_3y_4 + 40x_4y_4
\end{cases}$$

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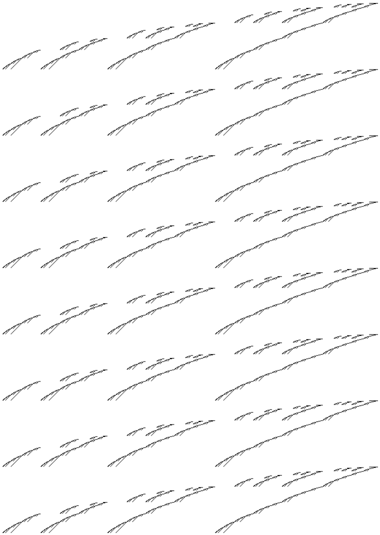
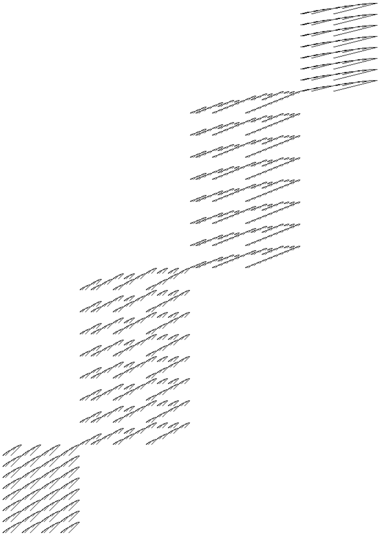
This system is bilinear (degree 1 in (x_1, x_2, x_3, x_4) and in (y_1, y_2, y_3, y_4)).

Macaulay matrices for the bilinear example



Affine Macaulay matrix at deg 5 of the example system

Macaulay matrices for the bilinear example



Affine Macaulay matrix at deg 5 of the example system
… after reordering the rows and columns

Algorithm and complexity

Algorithm: consider the signatures multi-degree by multi-degree

Complexity:

- Size of the matrices: $\binom{n_1 + d - 1}{d}$ rows, $\binom{n_2 + d - 1}{d}$ columns (with $n_1, n_2 \in \{n_x, n_y\}$)
- Degree of regularity?
- What is regularity???

Homogeneous case: $(f_1, \ldots, f_n)$ is a regular sequence

$\iff$ for all i, $qf_i \in \langle f_1, \ldots, f_{i-1} \rangle \implies q \in \langle f_1, \ldots, f_{i-1} \rangle$

$\iff$ $\mathrm{HS}(T) = \dfrac{(1 - T^{d_1}) \cdots (1 - T^{d_n})}{(1 - T)^n}$

$\iff$ $V(f_1, \ldots, f_m)$ has dimension 0

$\iff$ $V(f_1, \ldots, f_m) = \{0\}$

Bilinear case: (wrt the variables $x_1, \ldots, x_{n_x}$ and $y_1, \ldots, y_{n_y}$)

$V(f_1, \ldots, f_n)$ always contains $V(x_1, \ldots, x_{n_x})$ and $V(y_1, \ldots, y_{n_y})$

$\implies$ $f_1, \ldots, f_n$ cannot be a regular sequence!

Bi-regular sequence: $f_1, \dots, f_m$ is a bi-regular sequence iff for all i,

$$q f_i \in \langle f_1, \dots, f_{i-1} \rangle \implies \mathrm{LM}(q) \in \langle \mathcal{M}^{\boldsymbol{x}}_{i-n_y}(n_y) \rangle + \langle \mathcal{M}^{\boldsymbol{y}}_{i-n_x}(n_x) \rangle + \mathrm{LM}(\langle f_1, \dots, f_{i-1} \rangle)$$

where $\mathcal{M}^{\boldsymbol{v}}_k(d)$ is the set of monomials of degree d in $v_1, \dots, v_k$.

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Theorem 6.22. *Let $f_1, \dots, f_m \in R$ be a bi-regular bilinear sequence, with $m \leq n_x + n_y$. Then its Hilbert bi-series is*

$$\mathrm{mHS}_{\mathbb{K}[x_1,\dots,x_{n_x},y_1,\dots,y_{n_y}]/I}(t_1,t_2) = \frac{(1-t_1t_2)^m + N_m(t_1,t_2) + N_m(t_2,t_1)}{(1-t_1)^{n_x+1}(1-t_2)^{n_y+1}},$$

$$N_m(t_1,t_2) = \sum_{\ell=1}^{m-(n_y+1)} (1-t_1t_2)^{m-(n_y+1)-\ell} t_1 t_2 (1-t_2)^{n_y+1} \left[1 - (1-t_1)^\ell \sum_{k=1}^{n_y+1} t_1^{n_y+1-k} \binom{\ell+n_y-k}{n_y+1-k}\right]$$

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Degree of regularity: For a bi-regular bilinear sequence with $m = n_x + n_y$, $d_{\mathrm{reg}} \leq \max(n_x, n_y) + 2$ (not generically sharp)

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With higher-degree (bi-homogeneous) or more groups (multi-homogeneous), nothing is known.

Algorithm

In. $F = \{f_1, \ldots, f_m\}$ structured polynomials, $D \in \mathbb{N}$

Out. G Gröbner basis up to "structure degree" D

1. For d from 0 to D
   - 1.1 Form the Macaulay matrix M of "structure degree" d of the system (matrix whose rows are all the $m f_i$ for a choice of monomials m depending on d)
   - 1.2 Echelon-reduce the matrix M
   - 1.3 Add to G each polynomial corresponding to a reduced row
2. Return G

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Examples:

- Weighted-homogeneous: weighted-degree
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Examples:

- Weighted-homogeneous: weighted-degree
- Multi-homogeneous: multi-degree
- Group-invariant systems: G-degree (Faugère, Svartz 2012, 2013)
- Sparse systems: sparse degree (Faugère, Svartz, Spaenlehauer 2014, Faugère, Bender 2018...)
- ...

Definitions:

- a matrix of weights is a matrix $\boldsymbol{W} = (w_{i,j}) = \begin{pmatrix} W_1 \\ \vdots \\ W_k \end{pmatrix} \in \mathbb{Z}^{k \times n}$ with rank k
- the matrix-weighted degree of a monomial $\boldsymbol{X}^\alpha$ is

$$\deg_{\boldsymbol{W}}(\boldsymbol{X}^\alpha) = \boldsymbol{W} \cdot \alpha = (\deg_{W_1}(\boldsymbol{X}^\alpha), \dots, \deg_{W_k}(\boldsymbol{X}^\alpha))$$

- Matrix-weighted homogeneous polynomials and ideals are defined as usual

Examples:

- weighted homogeneous systems are matrix-weighted homogeneous with $k = 1$
- multi-homogeneous systems are matrix-weighted homogeneous with

$$W_i = (\underbrace{0, \dots\dots, 0}_{n_1+\dots+n_{i-1}}, \underbrace{1, \dots, 1}_{n_i}, \underbrace{0, \dots\dots, 0}_{n_{i+1}+\dots+n_k}).$$

Algorithm:

- use the previous algorithm following the matrix-weighted degree
- or use the change of variable $X_i \mapsto Y_{i,1}^{w_{1,i}} \cdots Y_{i,k}^{w_{k,i}}$ to recover a multi-homogeneous ideal, prune out the unnecessary (repeating) monomials, and use Spaenlehauer's algorithms
- the two strategies are equivalent

Complexity:

- Size of the matrices: number of solutions of linear diophantine equations (no closed form even in the weighted homogeneous case)
- Regular sequences, degree of regularity: unknown (cannot be less complicated than the multi-homogeneous case)
- It may be impossible to give a satisfying notion of dimension for the solutions (like the projective dimension for homogeneous systems)

Summary

- Overview of linear algebra algorithms for computing Gröbner bases for structured systems
- Examples: weighted homogeneous, multi-homogeneous, matrix-weighted homogeneous
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Aspects not discussed

- Effective elimination of reduction to zero (F5 criterion, extensions for the structures)
- Additional optimizations (e.g. parallelism)
- Number of solutions, FGLM algorithm
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- Under-determined case, over-determined case
- Sparse Gröbner bases

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Thank you for your attention!