Signature Gröbner bases and cofactor computation in the free algebra

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Question: Die Entscheidung ob die vorgelegte Grundform eine von 0 verschiedene Invariante besitzt oder nicht. [Hilbert 1893]

David Hilbert

Question: Given $f_1, \ldots, f_m, p \in K[X_1, \ldots, X_n]$, decide if $p \in \langle f_1, \ldots, f_m \rangle$. [Hilbert 1893]

Invariante besitzt oder nicht.

David Hilbert

The Ideal Membership Problem and Gröbner bases

Question: Given $f_1, \dots, f_m, p \in K[X_1, \dots, X_n]$, decide if $p \in \langle f_1, \dots, f_m \rangle$. [Hilbert 1893]

Polynomial system
$f_1, \dots, f_m$

↓

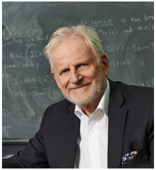
Buchberger	[Buchberger 1965]
F4	[Faugère 1999]
F5	[Faugère 2002]
FGLM	[Faugère Gianni Lazard Mora 1995]

↓

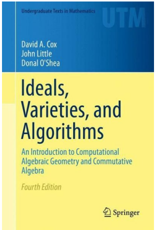
Gröbner basis G

↓

Reduction of p mod. G
+ zero test



Bruno Buchberger



Central in effective algebra and geometry

- List the solutions of a system
- Eliminate variables, compute projections
- Parametrization, implicitization
- Bases for differential operators, for word polynomials in the free algebra…
- Bases for modules

Setting:

- R field, $A = R\langle X_1, \ldots, X_n \rangle$ free algebra over R
- Monomials are words: $X_{i_1} X_{i_2} \cdots X_{i_d}$
- Monomial ordering and reduction are defined as usual
- Gröbner bases are defined as usual
- Application: proof of formulas
  "Does a relation follow from a prescribed set of axioms?"

What is not usual:

- The free algebra is not Noetherian
- Most ideals do not admit a finite Gröbner basis
- It is not decidable whether an ideal admits a finite Gröbner basis

Polynomials
$f_1, \dots, f_m$

Gröbner basis

Ideal Membership Problem

*"Does there exist (a_i)
such that
$p = a_1 f_1 + \cdots + a_m f_m$?"*

Polynomials
$f_1, \dots, f_m$

[Faugère 1999]

Gröbner basis

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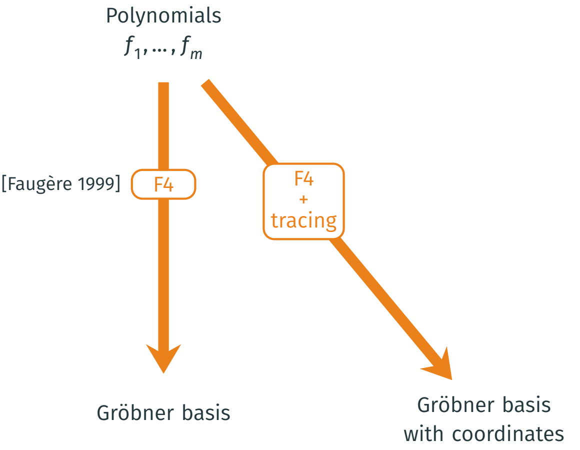
"Does there exist (a_i) such that $p = a_1 f_1 + \cdots + a_m f_m$?"

IMP with certificate

"Compute (a_i) such that $p = a_1 f_1 + \cdots + a_m f_m$"



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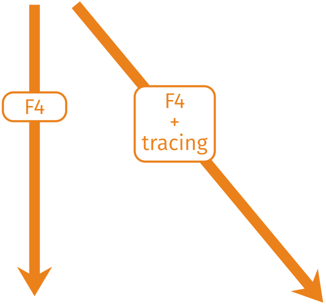

Gröbner basis

Gröbner basis
with coordinates

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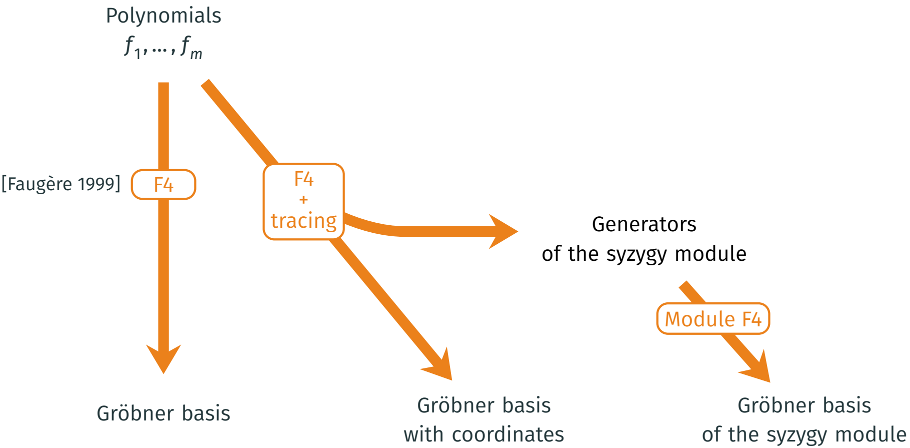
Module of syzygies

"Find all (a_i) such that $a_1 f_1 + \cdots + a_m f_m = 0$"

Why signature Gröbner bases?

Commutative case



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F5/GVW

[Faugère 2002]
[Gao Volny Wang 2010]

Gröbner basis
with signatures

Gröbner basis

Gröbner basis
with coordinates

Gröbner basis
of the syzygy module

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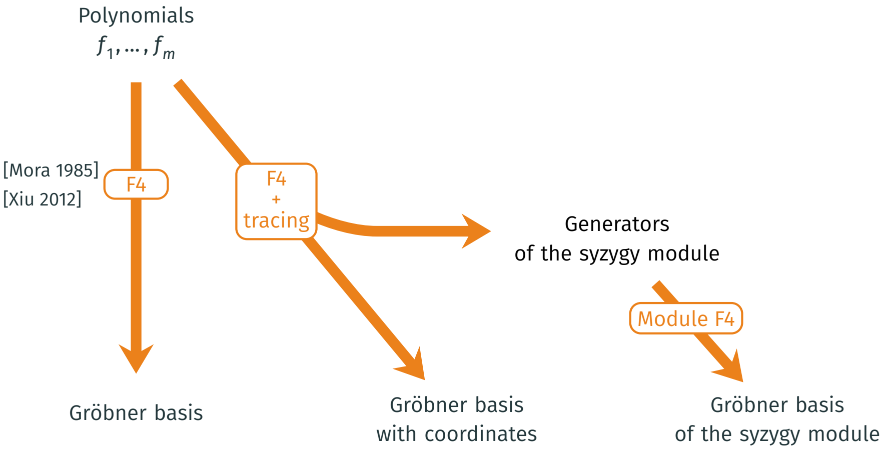

Ideal Membership Problem

"Does there exist (a_i, b_j) such that $p = a_1 f_1 b_1 + \dots + a_m f_m b_m$?"

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Polynomials
$f_1, \dots, f_m$

F5/GVW

Gröbner basis
with signatures

This work

- Algorithm for signature GB in the free algebra
- First algo. computing a GB of the module of syzygies

Gröbner basis

Gröbner basis
with coordinates

Gröbner basis
of the syzygy module

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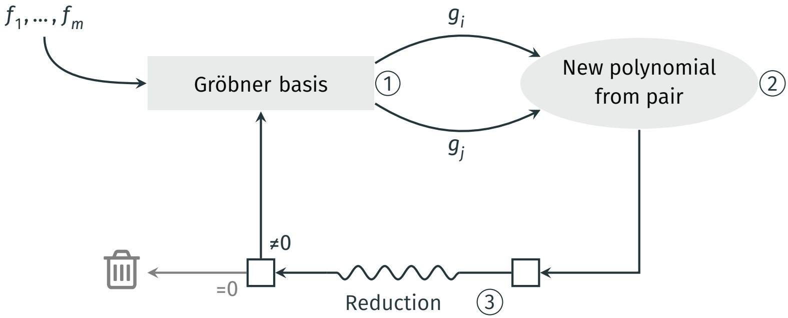

1. **Selection**: selection strategy
2. **Construction**: S-polynomials
3. **Reduction**

Signatures

Problem: useless computations: 🗑 ⟶ ♻

$p = p_1 f_1 + p_2 f_2 + \cdots + p_m f_m$

$q = q_1 f_1 + q_2 f_2 + \cdots + q_m f_m$

$p - q = 0?$

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- 1^{st} idea: keep track of the representation of the ideal elements
  [Möller, Mora, Traverso 1992]

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$\boldsymbol{p} - \boldsymbol{q} = \left(p_1 \boldsymbol{e}_1 + \cdots + p_m \boldsymbol{e}_m\right) - \left(q_1 \boldsymbol{e}_1 + \cdots + q_m \boldsymbol{e}_m\right)$

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- 2^{nd} idea: we do not need the full representation, the largest term is enough
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$= \mathrm{LT}(p_k)\boldsymbol{e}_k - \mathrm{LT}(q_l)\boldsymbol{e}_l$ + smaller terms

$= \mathrm{LT}(p_k)\boldsymbol{e}_k$ + smaller terms if $\mathrm{LT}(p_k)\boldsymbol{e}_k > \mathrm{LT}(q_l)\boldsymbol{e}_l$

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$\mathrm{sig}(\boldsymbol{p})$ = signature of $\boldsymbol{p}$

$p - q = 0?$

$$\boldsymbol{p} - \boldsymbol{q} = \left(p_1 \boldsymbol{e}_1 + \cdots + p_m \boldsymbol{e}_m\right) - \left(q_1 \boldsymbol{e}_1 + \cdots + q_m \boldsymbol{e}_m\right)$$
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Regular addition

Module framework

Setting:

- Input: $f_1, \dots, f_m \in A = R[\boldsymbol{X}]$ spanning the ideal I
- Module $M = A\boldsymbol{e}_1 \oplus \cdots \oplus A\boldsymbol{e}_m \simeq A^m$ with the map $M \to I$, $\boldsymbol{e}_i \mapsto f_i$
- Monomials in M are ordered with an ordering compatible with that on A
- Signature-polynomial pair: $(\boldsymbol{s}, f)$ with $f = \sum a_i f_i$ and $\boldsymbol{s} = \mathrm{LM}(\sum a_i \boldsymbol{e}_i)$
- Syzygy in M: $\boldsymbol{z} = \sum z_i \boldsymbol{e}_i \in M$ such that $\sum z_i f_i = 0$

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Regular operations:

- Multiplying a sig-poly pair by a term in A is easy
- We can only compute the result of regular additions: $(\boldsymbol{s}, f) + (\boldsymbol{t}, g) = (\max(\boldsymbol{s}, \boldsymbol{t}), f + g)$ if $\boldsymbol{s} \neq \boldsymbol{t}$
- We define regular S-polynomials and regular reductions in that way

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s-reductions: $(\mathrm{sig}(\boldsymbol{f}), f)$ s-reduces to $(\mathrm{sig}(\boldsymbol{h}), h)$ modulo $(\mathrm{sig}(\boldsymbol{g}), g)$ if:

- $t\mathrm{LT}(f) = \mathrm{LT}(f)$
- $h = f - tg$
- $t\mathrm{sig}(\boldsymbol{g}) \leq \mathrm{sig}(\boldsymbol{f})$

"A s-reduction doesn't increase the signature, a regular reduction doesn't change it."

Signature Gröbner bases

Signature Gröbner basis:

- set $\mathcal{G}$ of sig-poly pairs such that every sig-poly pair of M is s-reducible modulo $\mathcal{G}$
- Property: the polynomial parts of a S-GB form a Gröbner basis

Signature basis of syzygies:

- set $\mathcal{Z}$ of signatures such that every syzygy in M is reducible modulo $\mathcal{Z}$
- equivalently, generating set for the leading terms of the syzygies in M

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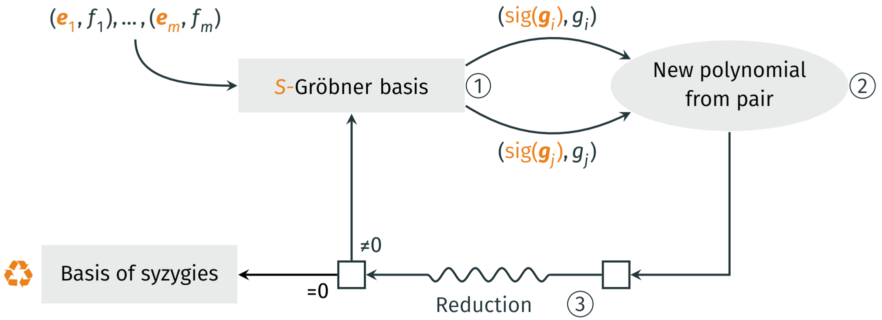
Buchberger's algorithm, with signatures and restricted to regular operations, computes both of those



1. **Selection**: non-decreasing signatures
2. **Construction**: regular S-polynomials
3. **Reduction** (regular)

Why do we care? 1: Criteria

Singular criterion

- if two regular-reduced elements have the same signature, they s-reduce each other
- Consequence: it is enough to add one of them
- Consequence: we can discard singular reducible elements after reduction

Syzygy criterion

- if $(\boldsymbol{s}, 0)$ is a sig-poly pair, any element with signature divisible by $\boldsymbol{s}$ regular-reduces to 0
- Consequence: we can discard such elements before computing the S-pol

F5 criterion

- $\mathrm{sig}(f_i \boldsymbol{e}_j - f_j \boldsymbol{e}_i) = \max\left(\mathrm{LM}(f_i)\boldsymbol{e}_j, \mathrm{LM}(f_j)\boldsymbol{e}_i\right)$ is the signature of a syzygy
- Consequence: we can add them to the basis of syzygies early

Theorem [Gao, Volny, Wang 2015]

Given $\mathcal{G}$ a signature Gröbner basis and $\mathcal{Z}$ a signature basis of syzygies, one can reconstruct:

- a Gröbner basis with coordinates $\mathcal{G}_{\text{full}}$;
- a Gröbner basis of the module of syzygies $\mathcal{Z}_{\text{full}}$.

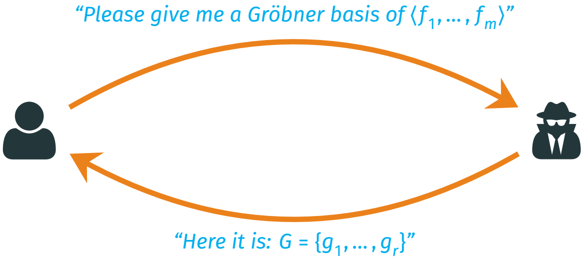
Reconstructing the module elements from the signatures

In
- $\mathcal{G} = \{(\boldsymbol{s}_i, g_i)\}$ a signature Gröbner basis
- $\mathcal{Z} = \{(\boldsymbol{z}_i, 0)\}$ a signature basis of syzygies

Out
- $\mathcal{G}_{\text{full}}$ a Gröbner basis with coordinates
- $\mathcal{Z}_{\text{full}}$ a Gröbner basis of the module of syzygies

1. $\mathcal{G}_{\text{full}} \leftarrow \{(\boldsymbol{e}_i, f_i) : i \in \{1, \dots, m\}\}$ (reducing if needed)
2. For $(\boldsymbol{s}_i, g_i) \in \mathcal{G}$ in increasing order of signatures, do
    2.1 Find $\boldsymbol{g}_j \in \mathcal{G}_{\text{full}}$ s.t. there exists a term t with $t\text{sig}(\boldsymbol{g}_j) = \boldsymbol{s}_i$ (and $t\text{LM}(\boldsymbol{g}_j)$ minimal)
    2.2 Perform regular reductions of $t\boldsymbol{g}_j$ by $\mathcal{G}_{\text{full}}$ until not reducible
    2.3 Add the result to $\mathcal{G}_{\text{full}}$
3. With $\mathcal{G}_{\text{full}}$ known, reconstruct $\mathcal{Z}_{\text{full}}$ in the same way

Remark: Certification of Gröbner basis computations



Problem: how to verify that G is a Gröbner basis of $I = \langle f_1, \dots, f_m \rangle$?

Two conditions:

1. G is a Gröbner basis (of $\langle G \rangle$):
   This can be tested by checking that all S-pols of G reduce to 0 (Buchberger's criterion)
2. $\langle G \rangle = I$, or $f_1, \dots, f_m \in G$ and $G \subset I$:
   This is as difficult as the ideal membership problem!

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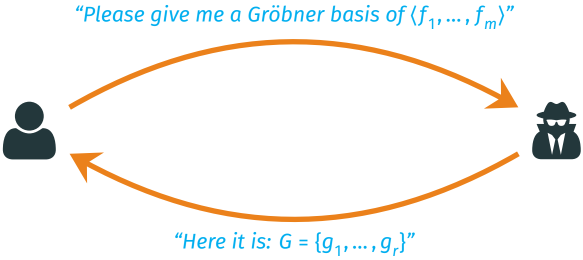

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If the server provides a signature Gröbner basis, testing condition 2 becomes easy

Question: can we make a certificate for condition 1 using signatures?

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- $\mathcal{Z} = \{(\boldsymbol{z}_i, 0)\}$ a signature basis of syzygies

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- $\mathcal{G}_{\mathrm{full}}$ a Gröbner basis with coordinates
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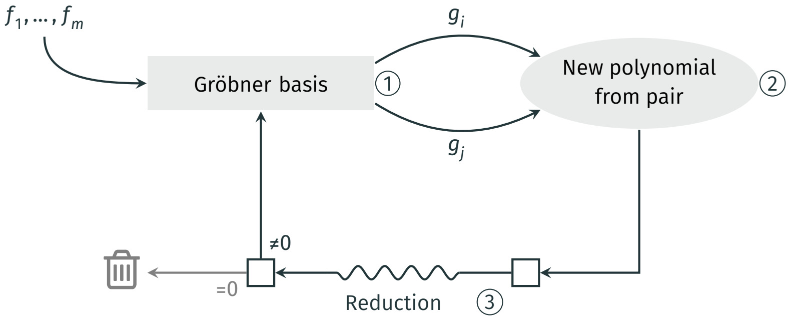
Non-commutative Buchberger's algorithm



1. **Selection**: fair selection strategy *"Every S-polynomial is selected eventually."*
2. **Construction**: S-polynomials
3. **Reduction**

Several ways to make S-polynomials

- **Overlap ambiguity**

- **Inclusion ambiguity**

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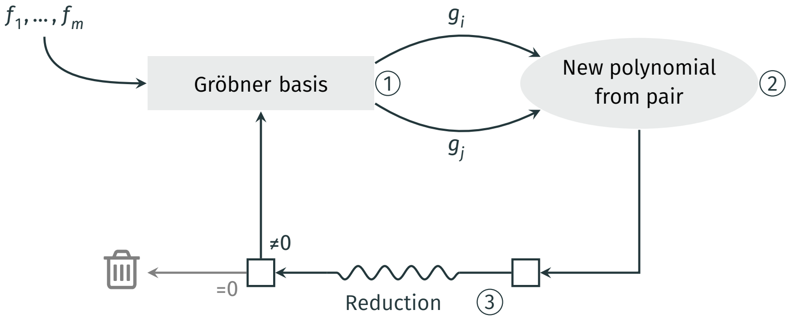
Remarks:

- The combination need not be minimal, and S-polynomials are not unique!
- $xyxy$ has an (overlap) ambiguity with itself: $xyxy$ / $xyxy$
- $xxyx$ and xy have two ambiguities: $xxyx$ / xy $xxyx$ / xy
- Two polynomials can only give rise to finitely many S-polynomials
- It is required that the central part is non-trivial (coprime criterion)



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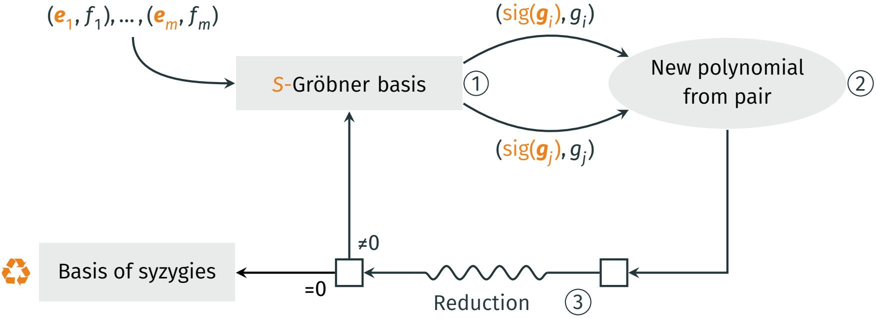
Non-commutative setting:

- Bimodule $M = A\boldsymbol{e}_1 A \oplus \cdots \oplus A\boldsymbol{e}_m A$ with the expected morphism $M \to A$ with image I
- Equipped with a module monomial ordering as before
- The ordering must additionally be fair (isomorphic to $\mathbb{N}$)
- Sig-poly pairs $(\boldsymbol{s}, f)$ with $f = \sum a_i f_i b_i$ and $\boldsymbol{s} = \mathrm{LM}(\sum a_i \boldsymbol{e}_i b_i)$
- Regular S-polynomials and reductions are defined as before



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Termination

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Termination: trivial syzygies

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Question 2: Okay, but what if they do?

Termination: trivial syzygies and how to find them

Question 1: Does the algorithm always terminate?

- Of course not, because most ideals do not have a finite Gröbner basis.

Question 2: Okay, but what if they do?

- Still not. In most cases, the module of syzygies does not have a finite Gröbner basis
- Conjecture: it's always the case if $n > 1$ (non-commutative) and $m > 1$ (non-principal)

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Obstruction: Trivial syzygies! [Hofstadler V. 2021] [Chenavier Léonard Vaccon 2021]

- Syzygies of the form $\boldsymbol{f} \blacksquare g - f \blacksquare \boldsymbol{g}$ for any monomial $\blacksquare$
- Signature: $\max\big(\mathrm{sig}(\boldsymbol{f}) \blacksquare \mathrm{LM}(g), \mathrm{LM}(f) \blacksquare \mathrm{sig}(\boldsymbol{g})\big)$
- Because $\blacksquare$ is put in the middle, this set is usually not finitely generated

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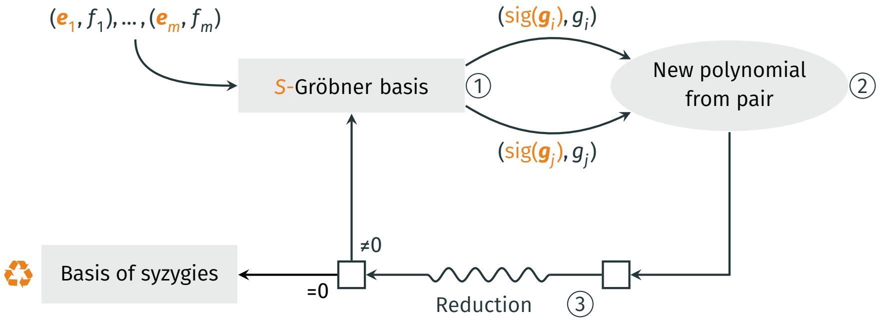
Solution: Signatures!

- Identifying trivial syzygies is what signatures were made for (F5 criterion)
- Not just an optimization, but necessary for termination for some ideals



1. **Selection**: non-decreasing signatures
2. **Construction**: regular S-polynomials which are not eliminated by the F5 criterion
3. **Reduction** (regular)

What do we get?

Output of the algorithm: a Gröbner basis with signatures, allowing to recover

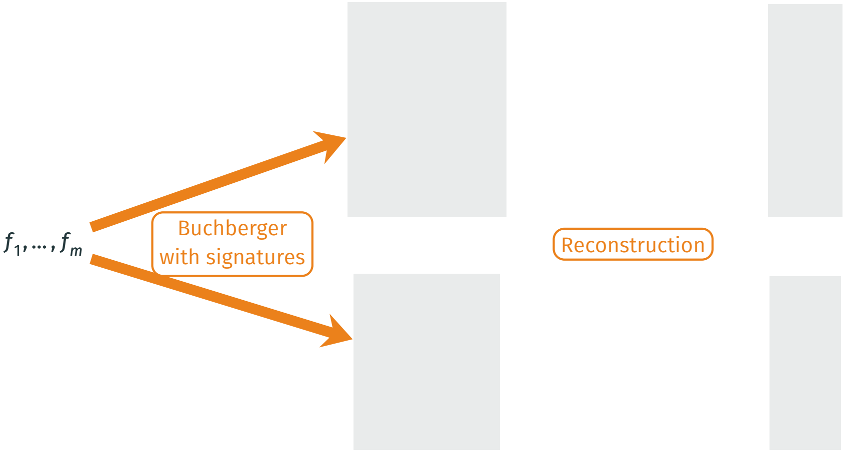
- a Gröbner basis $\mathcal{G}$ with the coordinates
- a set $\mathcal{H}$ of syzygies such that $\mathcal{H} \cup \{\text{trivial syzygies of } \mathcal{G}\}$ is a basis of the module of syzygies
- a way to test if any module monomial is the leading term of a syzygy

Results:

- The algorithm enumerates a signature Gröbner basis, by increasing order of signatures
- The algorithm terminates iff the ideal admits a finite signature Gröbner basis
- This implies that the ideal admits a finite GB and a finite "basis of non-trivial syzygies" $\mathcal{H}$
- **Conjecture**: the converse holds

This is the first algorithm producing an effective representation
of some modules of syzygies in the free algebra!

Reconstruction in the non-commutative case



- The reconstruction can work with partial output from Buchberger+signatures
- The reconstruction terminates with finite input

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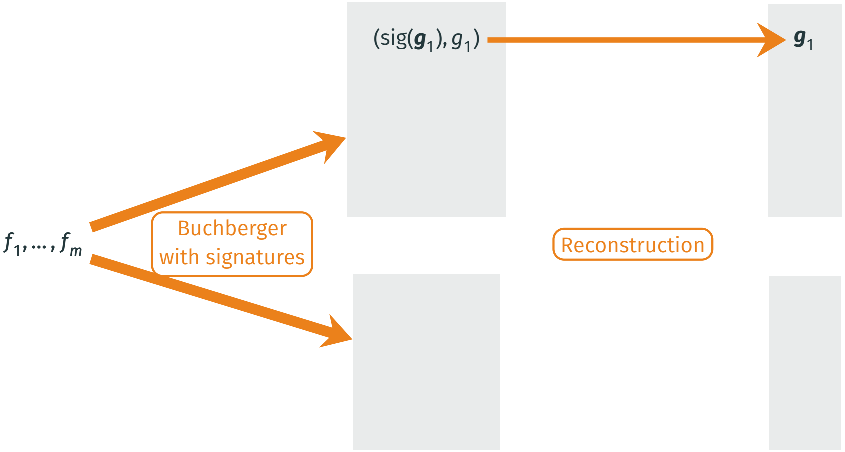

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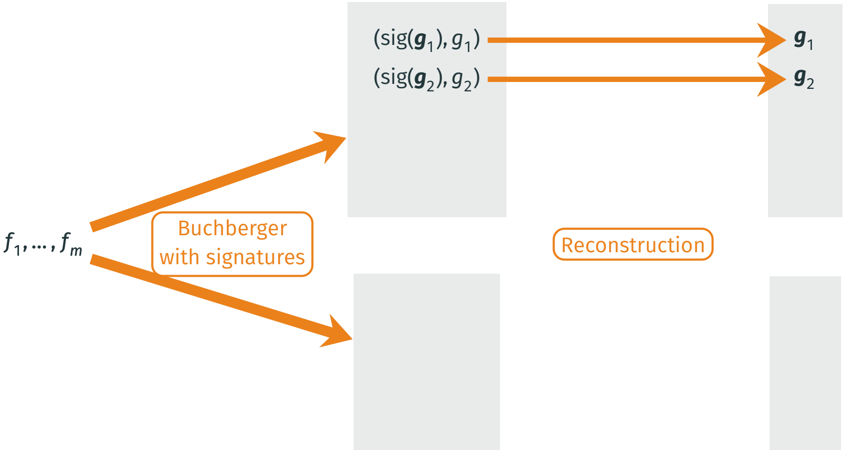

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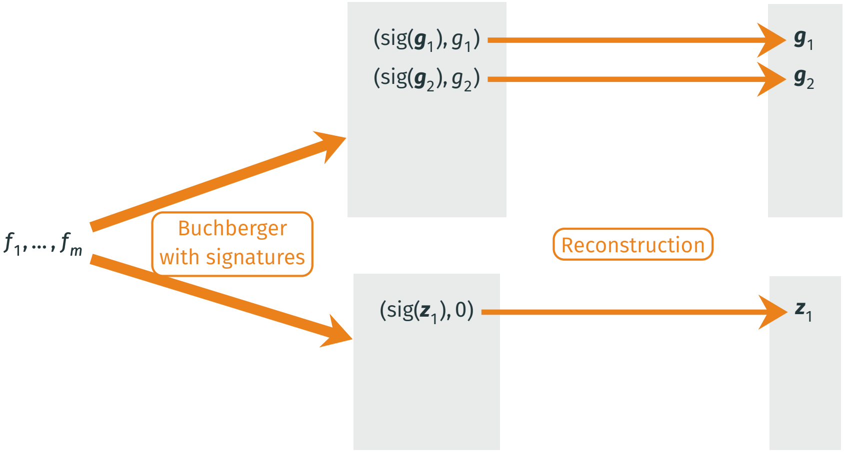

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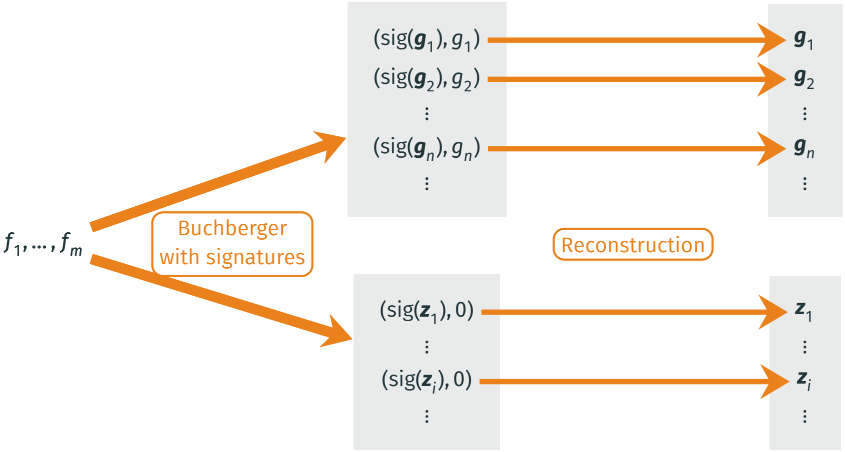

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Implementation

What we have

- Toy implementation in Mathematica
- Part of the package `OperatorGB`: `https://clemenshofstadler.com/software/`

Example	Signature			Buchberger			Buchberger + chain		
	S-poly	Red 0	Time	S-poly	Red 0	Time	S-poly	Red 0	Time
`lv2-100`	201	0	60	9702	4990	43	9702	4990	46
`tri1`	335	164	62	9435	8897	16	3480	3288	6

Remarks

- The F5 criterion is necessary to maximize the chances of the algorithm terminating
- The PoT ordering is not fair
- The F5 criterion is expensive! (quadratic in the size of $\mathcal{G}$)
- Reconstruction of the module representation can be very expensive
  (no bound on the rank of the tensors)

Conclusion

This work

- Signature-based algorithm enumerating signature Gröbner bases in the free algebra
- Terminates whenever a finite signature Gröbner basis exists
- Taking care of trivial syzygies is necessary for termination
- Effective and finite representation of the module of syzygies in some non-trivial cases

Open questions and future directions

- Conjecture on characterization of existence of finite signature Gröbner basis
- Use of signatures for the computation of short representations
- Computations in quotients of the algebra, elimination...

More details and references

- Hofstadler and Verron, *Signature Gröbner bases, bases of syzygies and cofactor reconstruction in the free algebra*, `ArXiV:2107.14675`

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Merci pour votre attention !