Integral bases of P-recursive sequences

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What does it mean to be integral?

Integral elements	All elements	
$\mathbb{Z}_p$	$\mathbb{Q}_p$	Local integrality
$\mathbb{C}[[x-\alpha]]$	$\mathbb{C}((x-\alpha))$	
$\mathbb{Z}$	$\mathbb{Q}$	Global integrality
$\mathbb{C}[x]$	$\mathbb{C}(x)$	

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$\mathbb{C}[x]$	Denom. $\in \mathbb{C}$	$\mathbb{C}(x)$	Global integrality
$\mathcal{O}_{\mathbb{C}[x]}$	Denom. of minpoly $\in \mathbb{C}$	$\overline{\mathbb{C}(x)}$	Global integrality

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"No pole" [Kauers Koutschan 2015]		$\left\{\begin{array}{c}\text{Sol. of diffeq} \\ \text{with poly. coefs.}\end{array}\right\}$	Global integrality

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"No pole" [Kauers Koutschan 2015]		$\left\{ \begin{array}{c} \text{Sol. of diffeq} \\ \text{with poly. coefs.} \end{array} \right\}$	
"val ≥ 0": **this work**		$\left\{ \begin{array}{c} \text{Sol. of rec.} \\ \text{with poly. coefs.} \end{array} \right\}$	

What does it mean to be integral?

Integral elements		All elements	Valuation at...
$\mathbb{Z}_p$	Denom. $\in \{\pm 1\}$	$\mathbb{Q}_p$	p prime
$\mathbb{C}[[x-\alpha]]$	Denom. $\in \mathbb{C}$	$\mathbb{C}((x-\alpha))$	$x-\alpha$
$\mathbb{Z}$	Denom. $\in \{\pm 1\}$	$\mathbb{Q}$	All primes
$\mathbb{C}[x]$	Denom. $\in \mathbb{C}$	$\mathbb{C}(x)$	All $x-\alpha$
$\mathcal{O}_{\mathbb{C}[x]}$	Denom. of minpoly $\in \mathbb{C}$ $\iff$ "no pole"	$\overline{\mathbb{C}(x)}$	All $x-\alpha$
"No pole" [Kauers Koutschan 2015]		$\left\{\begin{matrix}\text{Sol. of diffeq} \\ \text{with poly. coefs.}\end{matrix}\right\}$	All $x-\alpha$
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"val ≥ 0": **this work**

Local integrality

Global integrality

Better framework: integral operators

Polynomial algebras:

- "Algebraic equations": $C(x)[y]$, commutative: $xy = yx$

Algebraic case (finite extension):

- Given $\alpha(x) \in \overline{C(x)}$ or equivalently given $P \in C[x][y]$
- Question: What are integral elements of $C(x)(\alpha) = C(x)[y]/\langle P \rangle$?
- Answer: Q is integral iff for all $\alpha(x)$ sol of P, $(Q(\alpha))(x)$ does not have any pole
- Integral elements form a $C[x]$-algebra in $C(x)[y]$

Can we compute a basis of that set as a $C[x]$-module?

- Yes: Trager's algorithm, van Hoeij's algorithm
- Application: computation of integrals [Trager 1984]

Better framework: integral operators

Polynomial and Ore algebras:

- "Algebraic equations": $C(x)[y]$, commutative: $xy = yx$
- "Differential equations": $C(x)\langle D\rangle$, non-commutative: $Dx = xD + 1$

Differential case:

- Given $L \in C[x]\langle D\rangle$
- Question: What are integral elements of $C(x)\langle D\rangle / \langle L\rangle$?
- Answer: B is integral iff for all $\alpha(x)$ sol of L, $(B \cdot \alpha))(x)$ does not have any pole
- Integral elements form a $C[x]$-module in $C(x)\langle D\rangle$

Can we compute a basis of that $C[x]$-module?

- Yes: adaptation of van Hoeij's algorithm [Kauers, Koutschan 2015]
- Application: computation of integrals [Chen, van Hoeij, Kauers, Koutschan 2018]

Better framework: integral operators

Polynomial and Ore algebras:

- "Algebraic equations": $C(x)[y]$, commutative: $xy = yx$
- "Differential equations": $C(x)\langle D\rangle$, non-commutative: $Dx = xD + 1$
- "Recurrence equations": $C(x)\langle S\rangle$, non-commutative: $Sx = (x+1)S$

Recurrence case:

- Given $L \in C[x]\langle S\rangle$
- Question: What are integral elements of $C(x)\langle S\rangle/\langle L\rangle$?
- Answer: B is integral iff for all $\alpha(x)$ sol of L, $(B \cdot \alpha))(x)$ ... ???

Better framework: integral operators

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Recurrence case:

- Given $L \in C[x]\langle S\rangle$
- Question: What are integral elements of $C(x)\langle S\rangle/\langle L\rangle$?
- Answer: B is integral iff for all $\alpha(x)$ sol of L, $(B\cdot\alpha))(x)$ has "val" ≥ 0 everywhere
- Integral elements form a $C[x]$-module in $C(x)\langle S\rangle$

Can we compute a basis of that $C[x]$-module?

- Yes: adaptation of van Hoeij's algorithm [Chen, Du, Kauers, V. 2020]
- Application: computation of sums?

Local algorithm:

Input. $L \in C[x]\langle D\rangle$ with order r, $\alpha \in C$

Output. $B_1, \ldots, B_r$ basis of $C(x)\langle D\rangle/\langle L\rangle$ integral at α

1. $B_1, \ldots, B_r \leftarrow$ basis of $C(x)\langle D\rangle/\langle L\rangle$
2. For $d \in \{1, \ldots, r\}$:
3. While B_i is not integral at α
4. $B_i \leftarrow (x-\alpha)B_i$
5. While there exists $a_1, \ldots, a_{d-1} \in C$ such that $A := \frac{1}{x-\alpha}(a_1B_1 + \cdots + a_{d-1}B_{d-1} - B_d)$ is integral at α
6. $B_d \leftarrow A$
7. Return $B_1, \ldots, B_r$

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1. $B_1, \ldots, B_r \leftarrow$ basis of $C(x)\langle D\rangle/\langle L\rangle$ $\qquad$ $B_1, \ldots, B_r \leftarrow 1, \ldots, D^{r-1}$
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Van Hoeij's algorithm for finding integral bases (differential case)

$f_1, \ldots, f_r \in C((x-\alpha))$ basis of solutions of L

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2. For $d \in \{1, \ldots, r\}$:
3. While B_i is not integral at α — $B_i \cdot f_j$ has a pole at α for some j
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6. $B_d \leftarrow A$
7. Return $B_1, \ldots, B_r$

$$\iff \forall j,\ (a_1 B_1 + \cdots + a_{d-1} B_{d-1} - B_d) \cdot f_j(\alpha) = 0$$
$$\iff \forall j,\ a_1 B_1 \cdot f_j(\alpha) + \ldots a_{d-1} B_{d-1} \cdot f_j(\alpha) = B_d \cdot f_j(\alpha)$$

Linear system of equations

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$f_1, \dots, f_r \in C((x-\alpha))$
basis of solutions of L

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Linear system of equations

Global algorithm: loop over all $\alpha \in C$

Van Hoeij's algorithm for finding integral bases (differential case)

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Linear system of equations

Global algorithm: loop over all $\alpha \in C$

Only finitely many (roots of the leading coefficient of L)

Finding solutions of P-recursive sequences

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Natural action: L acts on $\mathbb{C}^{\mathbb{Z}}$ via $(n \cdot u)_k = ku_k$, $(S \cdot u)_k = u_{k+1}$

If $(u_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is a solution, then for all $n \in \mathbb{Z}$:

- $(n-1)u_{n+3} = -(n-3)u_{n+1} - (n-1)(n+1)u_n$

$\cdots$	-1	0	1	2	3	4	$\cdots$
		1	0	0			
		0	1	0			
		0	0	1			

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		0	1	0	0		
		0	0	1	-3		

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		1	0	0	-1	$\times$	$0 = -2$
		0	1	0	0	0	$0 = 0$
		0	0	1	-3	$\times$	$0 = -6$

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		0	0	0	0	1	

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		0	1	0	0	0	$0 = 0$
		0	0	1	-3	$\times$	$0 = -6$
		0	0	0	0	1	
		3	0	-1	0	0	$0 = 0$

Finding solutions of P-recursive sequences

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

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		1	0	0	-1	$\times$	$0 = -2$
		0	1	0	0	0	$\cdots$
		0	0	1	-3	$\times$	$0 = -6$
		0	0	0	0	1	$\cdots$
		3	0	-1	0	0	$\cdots$

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- $(n-1)(n+1)u_n = -(n-3)u_{n+1} - (n-1)u_{n+3}$

$\cdots$	-1	0	1	2	3	4	$\cdots$
		1	0	0	-1	$\times$	$0 = -2$
$0 = 4$	$\times$	0	1	0	0	0	$\cdots$
		0	0	1	-3	$\times$	$0 = -6$
	0	0	0	0	0	1	$\cdots$
$0 = 10$	$\times$	3	0	-1	0	0	$\cdots$

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$\cdots$	-1	0	1	2	3	4	$\cdots$
		1	0	0	-1	$\times$	$0 = -2$
$0 = 4$	$\times$	0	1	0	0	0	$\cdots$
		0	0	1	-3	$\times$	$0 = -6$
	0	0	0	0	0	1	$\cdots$
$0 = 10$	$\times$	3	0	-1	0	0	$\cdots$
	1	0	0	0	0	0	$\cdots$
$0 = 0$	0	6	-5	2	0	0	$\cdots$

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- $(n-1)(n+1)u_n = -(n-3)u_{n+1} - (n-1)u_{n+3}$

$\cdots$	-1	0	1	2	3	4	$\cdots$
		1	0	0	-1	$\times$	$0=-2$
$0=4$	$\times$	0	1	0	0	0	$\cdots$
		0	0	1	-3	$\times$	$0=-6$
$\cdots$	0	0	0	0	0	1	$\cdots$
$0=10$	$\times$	3	0	-1	0	0	$\cdots$
$\cdots$	1	0	0	0	0	0	$\cdots$
$\cdots$	0	6	-5	2	0	0	$\cdots$

Finding robust solutions of P-recursive sequences

$L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Deformed action: L acts on $\mathbb{C}(q)^{\mathbb{Z}}$ or $\mathbb{C}((q))^{\mathbb{Z}}$ via $(n \cdot u)_k = (k+q)u_k$
- Recover usual solutions by setting $q = 0$

0	1	2	3	4	5	$\cdots$
1	0	0				
0	1	0				
0	0	1				

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0	1	2	3	4	5	$\cdots$
1	0	0	$-q-1$			
0	1	0	0			
0	0	1	$\frac{3-q}{q-1}$			

Finding robust solutions of P-recursive sequences

$L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Deformed action: L acts on $\mathbb{C}(q)^{\mathbb{Z}}$ or $\mathbb{C}((q))^{\mathbb{Z}}$ via $(n \cdot u)_k = (k+q)u_k$
- Recover usual solutions by setting $q = 0$

0	1	2	3	4	5	$\cdots$
1	0	0	$-q-1$	$\frac{(q+1)(q-2)}{q}$		
0	1	0	0	$-q-2$		
0	0	1	$\frac{3-q}{q-1}$	$\frac{(q-3)(q-2)}{q(q-1)}$		

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q	0	0	$-q(q+1)$	$(q+1)(q-2)$	$(q-2)(1-q)$	...
0	0	q	$\frac{q(3-q)}{q-1}$	$\frac{(q-3)(q-2)}{q-1}$	$\frac{6+\dots}{q+1}$	...

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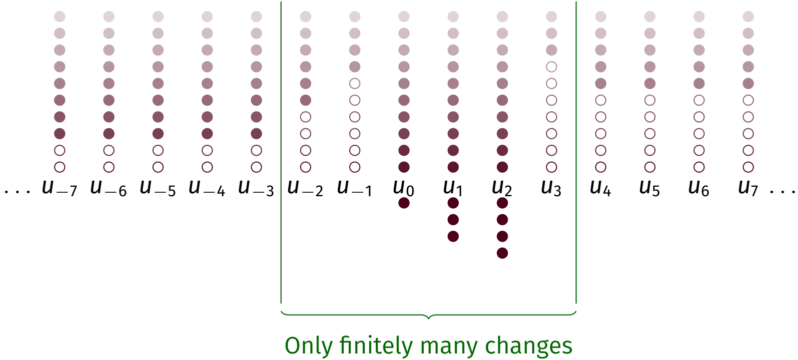
0	0	1	$\frac{3-q}{q-1}$	$\frac{(q-3)(q-2)}{q(q-1)}$	$\frac{6+\dots}{q(q+1)}$	⋯
q	0	0	$-q(q+1)$	$(q+1)(q-2)$	$(q-2)(1-q)$	⋯
0	0	q	$\frac{q(3-q)}{q-1}$	$\frac{(q-3)(q-2)}{q-1}$	$\frac{6+\dots}{q+1}$	⋯
$\frac{q-3}{q-1}$	0	$-q-1$	0	0	$(q+1)(q+3)$	⋯

P-recursive sequences: what are poles?

In practice, solutions of an operator look like this:



- Given $L \in C[n]\langle S\rangle$ with order r, it has r independent normalized solutions $u^{(1)}, \dots, u^{(r)}$ in $C((q))^{\mathbb{Z}}$
- $B \in C(n)\langle S\rangle/\langle L\rangle$ acts on those solutions
- Valuation of B at $\alpha \in \mathbb{Z}$: min of the valuations of $B \cdot u^{(i)}$ at α
- B is integral iff it has non-negative valuation everywhere

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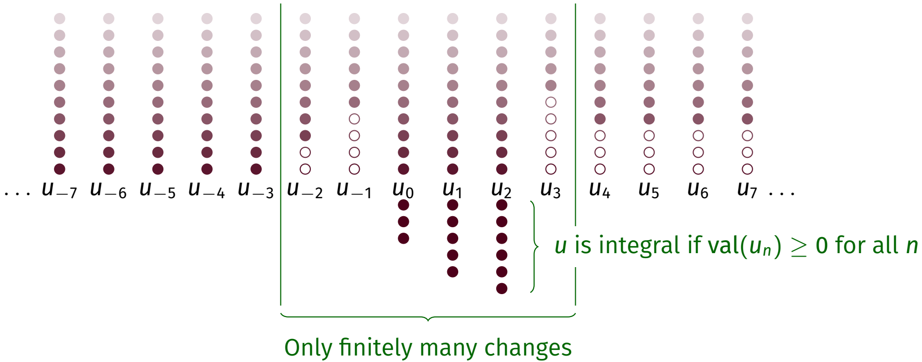

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Van Hoeij's algorithm for finding integral bases (recurrence case)

$u^{(1)}, \dots, u^{(r)} \in C((q))^{\mathbb{Z}}$
basis of solutions of L

Local algorithm: exactly the same!

Input. $L \in C[x]\langle S\rangle$ with order r, $\alpha \in \mathbb{Z}$

Output. $B_1, \dots, B_r$ basis of $C(x)\langle S\rangle/\langle L\rangle$ integral at α

1. $B_1, \dots, B_r \leftarrow$ basis of $C(x)\langle S\rangle/\langle L\rangle$ — $B_1, \dots, B_r \leftarrow 1, \dots, S^{r-1}$
2. For $d \in \{1, \dots, r\}$:
3. While B_i is not integral at α — $B_i \cdot u^{(j)}$ has val < 0 at α for some j
4. $B_i \leftarrow (x - \alpha)B_i$
5. While there exists $a_1, \dots, a_{d-1} \in C$
   such that $A := \frac{1}{x-\alpha}(a_1B_1 + \dots + a_{d-1}B_{d-1} - B_d)$ is integral at α
6. $B_d \leftarrow A$
7. Return $B_1, \dots, B_r$

$\iff \forall j, (a_1B_1 + \dots + a_{d-1}B_{d-1} - B_d) \cdot u^{(j)}$ has val > 0 at α

Linear system of equations

Global algorithm: loop over all $\alpha \in \mathbb{Z}$

Only finitely many
(between the roots of the leading and trailing coefficients of L)

Van Hoeij's algorithm for recurrences on an example (1)

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$

Basis of solutions:

	0	1	2	3	4	5	$\cdots$
u	1	0	0	$-1+O(q)$	$-2q^{-1}+O(1)$	$-2q^{-1}+O(1)$	$\cdots$
v	0	1	0	0	$-2+O(q)$	$-2+O(q)$	$\cdots$
w	0	0	1	$-3+O(q)$	$-6q^{-1}+O(1)$	$-6q^{-1}+O(1)$	$\cdots$

Van Hoeij's algorithm for recurrences on an example (2)

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$, $\alpha = 3$

B	1	
$(B \cdot u)_3$	$-1+O(q)$	
$(B \cdot v)_3$	0	
$(B \cdot w)_3$	$-3+O(q)$	

Van Hoeij's algorithm for recurrences on an example (2)

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$, $\alpha = 3$

B	1	S
$(B \cdot u)_3$	$-1+O(q)$	$-2q^{-1}+O(1)$
$(B \cdot v)_3$	0	$-2+O(q)$
$(B \cdot w)_3$	$-3+O(q)$	$-6q^{-1}+O(1)$

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$(B \cdot v)_3$	0	$-2+O(q)$	$-2q+O(q^2)$	$-2q+O(q^2)$
$(B \cdot w)_3$	$-3+O(q)$	$-6q^{-1}+O(1)$	$-6+O(q)$	$3q+O(q^2)$

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$, $\alpha = 3$

B	1	S	$(n-3)S$	$(n-3)S-2$	$S - \frac{2}{n-3}$
$(B\cdot u)_3$	$-1+O(q)$	$-2q^{-1}+O(1)$	$-2+O(q)$	$q+O(q^2)$	$1+O(q)$
$(B\cdot v)_3$	0	$-2+O(q)$	$-2q+O(q^2)$	$-2q+O(q^2)$	$-2+O(q)$
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What have we done?

- Definition of integral bases for P-recursive sequences
- Generalization of Van Hoeij's algorithm for computing them
- Implementation in the SageMath package `ore_algebra`

Why do we care?

- In the differential case, integral bases can be used to compute integrals
- We hope that in the recurrence case, they can be used to compute sums
- Future work: s/hope/prove/

What if it is the wrong definition for that?

- The definitions and the algorithm generalize to valued vector spaces
- No particularly restricting hypothesis
- So if the definition is wrong, we only have to find the correct one!

Thank you for your attention!

References

- Kauers and Koutschan, 'Integral D-finite Functions' (2015)
- Chen, van Hoeij, Kauers and Koutschan, 'Reduction-based Creative Telescoping for Fuchsian D-finite Functions' (2018)
- Chen, Du, Kauers and Verron, 'Integral P-Recursive Sequences' (2020)