On FGLM algorithms over Tate algebras

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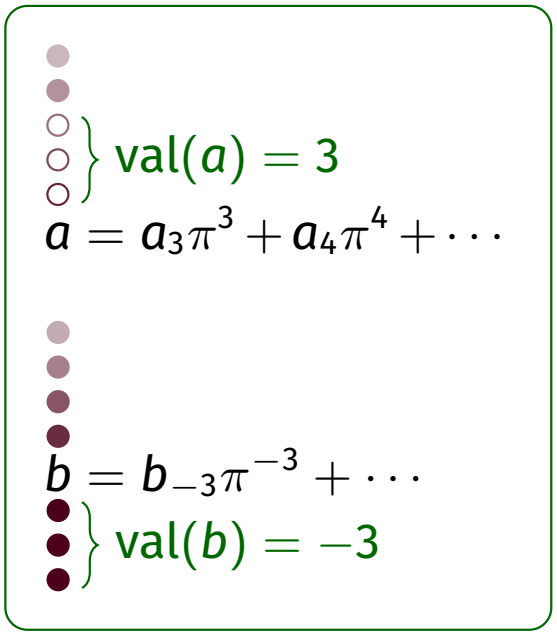
Setting and definitions

Valued field, valuation ring

▶ Field with a valuation $\mathrm{val} : K \to \mathbb{Z} \cup \infty$	$\mathbb{Q}_p$	$k((X))$
▶ Integer ring $K^\circ = \{x : \mathrm{val}(x) \geq 0\}$	$\mathbb{Z}_p$	$k[\![X]\!]$
▶ Uniformizer π s.t. $\pi K^\circ = \{x : \mathrm{val}(x) \geq 1\}$	p	X



Metric and topology

- "a is small" $\iff$ "$\mathrm{val}(a)$ is large"
- Non-archimedean metric: "small + small = small"
- $\mathbb{Q}_p, \mathbb{Z}_p, k((X)), k[\![X]\!]$ are complete for that topology

Rigid geometry and Tate series

- "Algebraic geometry, analytic geometry" bridge for non-archimedean geometry
- Main object: Tate series

Tate series

Definitions

$\mathbf{r} \in \mathbb{Q}^n$: convergence (log)-radii

- Tate algebra $K\{X_1, \ldots, X_n; r_1, \ldots, r_n\} = K\{\mathbf{X}; \mathbf{r}\}$
- Set of series $\sum_{\alpha \in \mathbb{N}^n} a_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ with $\mathrm{val}(a_\alpha) - \sum r_j \alpha_j \to \infty$
- *"Convergent for substitutions with* $\mathrm{val}(x_i) \geq -r_i$*"*
- smaller $r_i \iff$ smaller convergence radius $\iff$ larger algebra
- Convention: $r_i = \infty$ if finitely many terms in X_i (polynomial)

Examples

- Polynomials are Tate series for all radii (finite sums)
- $f = \sum_{i,j=0}^{\infty} \pi^i X^i = 1 + \pi X + \pi^2 X^2 + \pi^3 X^3 + \cdots$
  - $f \in K\{X\} = K\{X; 0\}$
  - $f \notin K\{X; 1\}$: for all terms, $\mathrm{val}(\pi^\alpha) - \alpha = 0 \not\to \infty$
- If $K = \mathbb{Q}_p$, $\exp(X)$ is a Tate series with $r < \frac{1}{p-1}$

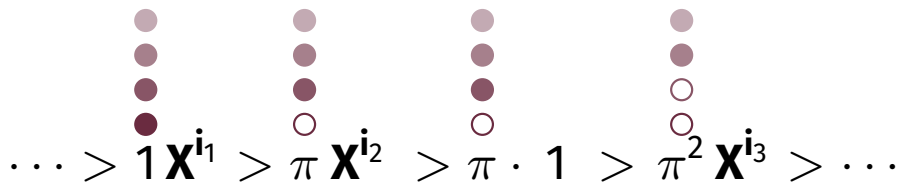
Gröbner bases over Tate algebras

Gröbner bases

- Multi-purpose tool for ideal arithmetic in polynomial algebras
- Ex: membership testing, elimination, intersection...
- Uses successive (terminating) reductions
- Requires the definition of a term ordering

Construction for Tate series

- Term ordering compatible with the topology
- First compare $\mathrm{val}(a_\alpha) - \sum r_j \alpha_j$ and break ties with a monomial order



- Non-terminating but convergent reduction (+ precision bound)
- Allows to use usual algorithms (Buchberger, F4) to compute Gröbner bases

Complexity bottleneck: reductions

Cost of reductions

- Not unusual with Gröbner bases
- Tate case: reductions are interrupted at the precision bound
- The cost grows badly with the precision
- **Question:** can we do better?

Possible improvements?

- Avoid useless reductions to zero
- Speed-up interreductions
- Exploit overconvergence

Series converging faster, *i.e.*, living in a smaller Tate algebra
Ex: polynomials (log-radii ∞) seen as Tate series

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→ In dim. 0: with FGLM **[this work]**

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Change of ordering and the FGLM algorithm

Change of ordering

- Useful in the classical case for two-steps strategies
- For zero-dimensional ideals, can be done efficiently with the FGLM algorithm [Faugère, Gianni, Lazard, Mora 1993]
- Complexity cubic (or faster) in the degree (number of solutions)

For Tate algebras

- Change of term ordering (monomial ordering and convergence radii)
- Complexity cubic in the degree, quasi-linear in the precision
- Can reduce partially reduced bases

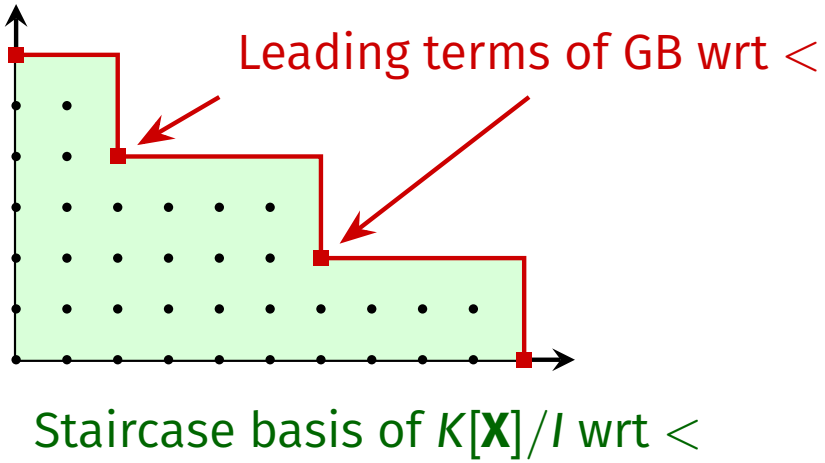
Idea for overconvergence

1. Compute a Gröbner basis in the smaller Tate algebra
2. Use change of ordering to transfer to the larger one

FGLM algorithm

Zero-dimensional ideal in $K[\mathbf{X}]$

- Finitely many solutions
- $K[\mathbf{X}]/I$ has finite dimension δ as a K-vector space
- Key object: matrices of $P \mapsto X_i P \bmod I$



Algorithm FGLM

G_1 reduced GB of $I \subset K[\mathbf{X}]$ wrt the order $<_1$

Mult. matrices by $X_1, \ldots, X_n$ in staircase B_1

G_2 reduced GB of $I \subset K[\mathbf{X}]$ wrt the order $<_2$

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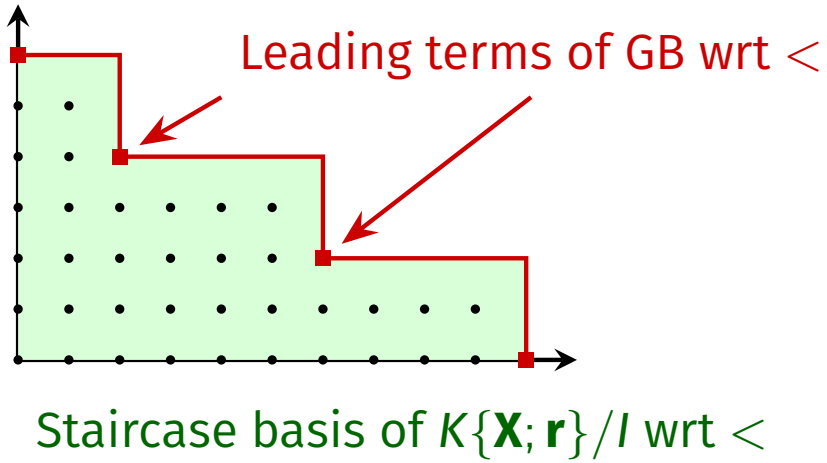

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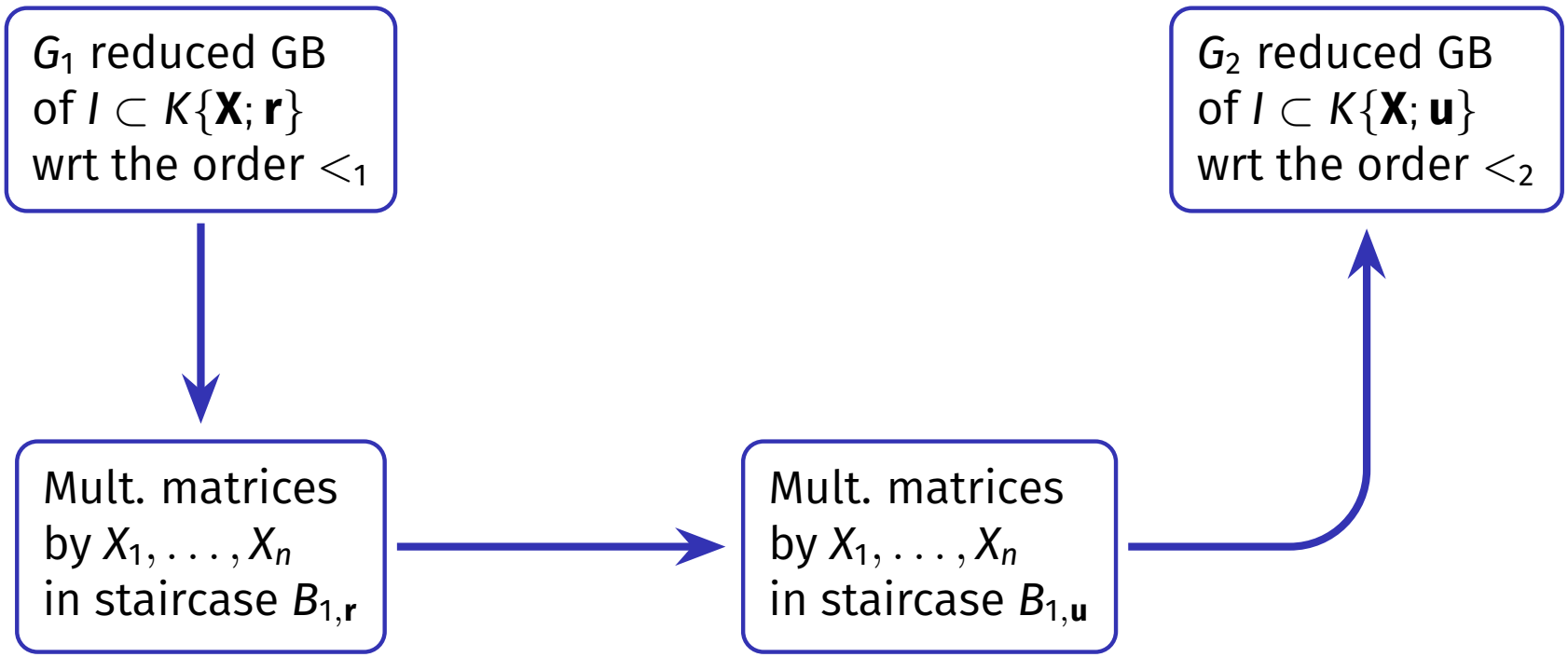
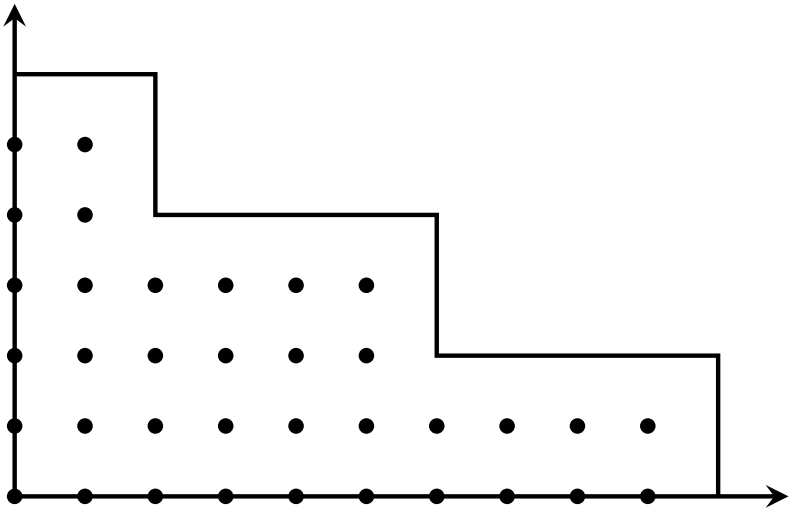


Iterative computation of the multiplication matrices

$\mathbf{r} = (0, \ldots, 0)$

- **Idea**: need to compute $\mathrm{NF}(X_i m)$ for all $i \in \{1, \ldots, n\}, m \in B$
- Proceed in increasing order and reuse the computations



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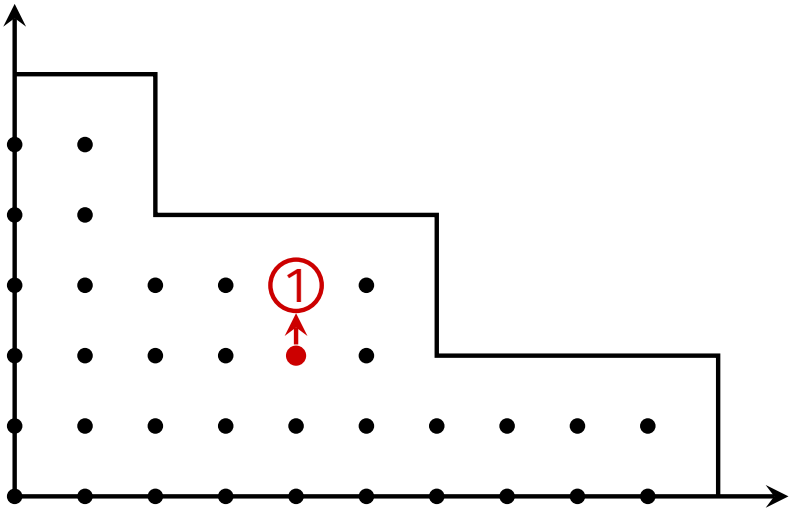

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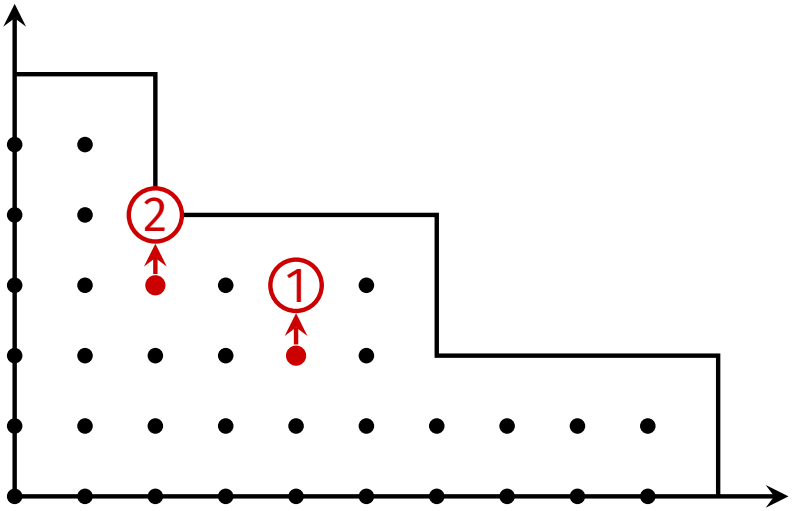

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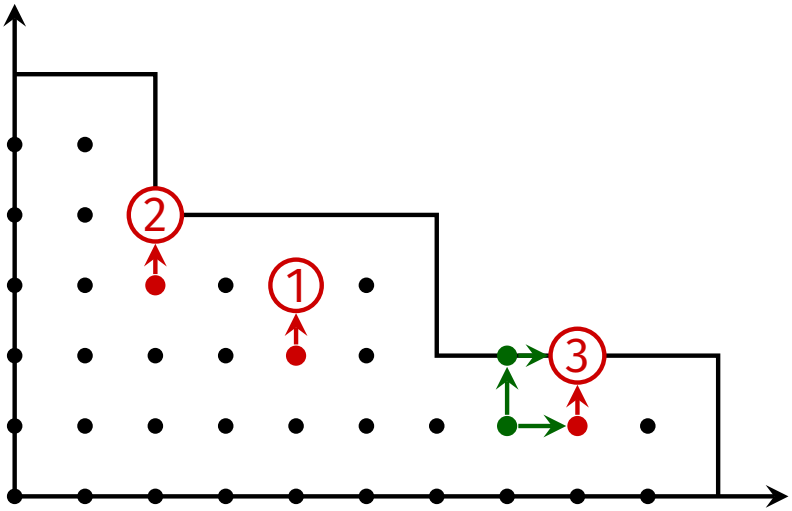

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3. Otherwise, write $m = X_j m'$ with
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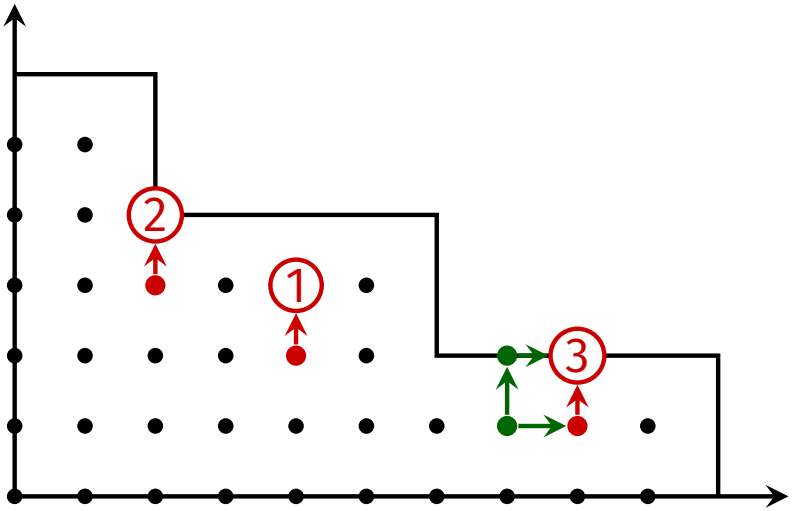

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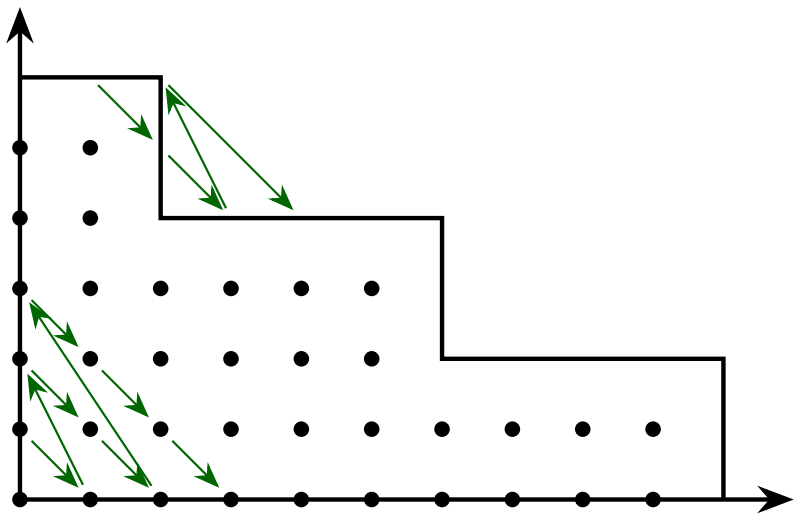
Why does it work?

- Usual case: $\mathrm{NF}(m)$ only involves monomials smaller than m
- Tate case: not true, but if not their coefficient is smaller than 1 (i.e. divisible by π)
- So we can recover the value mod π, and repeating k times, the value mod π^k:

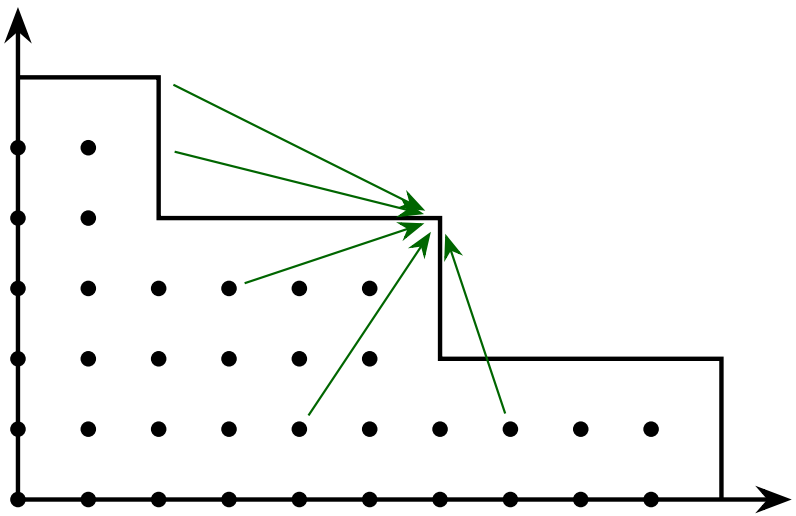
Improvements on the computation of the multiplication matrices



Incremental algorithm

- Follows the monomial ordering
- Cubic in δ, quadratic in precision
- Fast arithmetic does not help!

Improvements on the computation of the multiplication matrices



Recursive algorithm

- Query digits of the coefs as needed
- Functionally equivalent to incr. algo.
- Cubic in δ, quadratic in precision
- Order-agnostic (e.g. for other radii)

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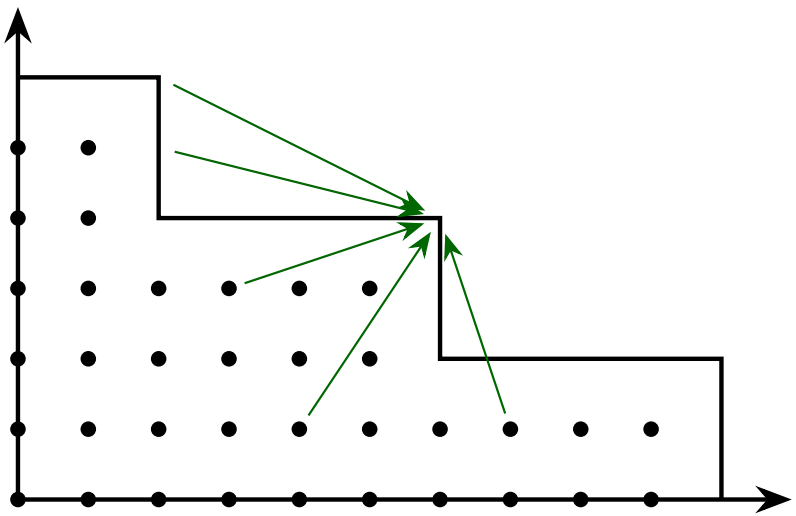

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Relaxed algorithm for $\mathbb{Q}_p$ or $\mathbb{Q}((X))$

- Lazy representation of objects
  + recursive definition
- Amortized log cost for each digit
- Complexity quasi-linear in precision

[v. d. Hoeven 1997] [Berthomieu, Lebreton 2012]
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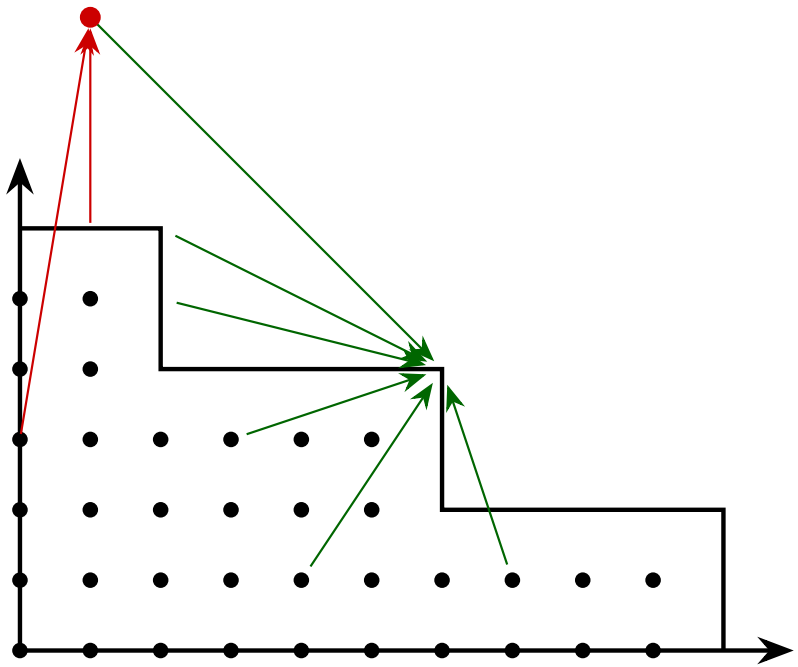

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What about non-reduced bases?

- We may need elements out of the staircase
- If reduced mod π, their coefficient is div. by π
- The relaxed algorithm still works!
- Complexity quasi-linear in precision, but unbounded in δ

Changing convergence radii: what happens to the staircase?

Example with $K = \mathbb{Q}_p$

- $K[x, y]$: $\mathbf{r} = (\infty, \infty)$
- $K\{x, y\}$: $\mathbf{u} = (0, 0)$

- $I = \langle px^2 - y^2, py^3 - x \rangle$
- $J = \langle y^2 - px^2, x - py^3 \rangle$
- $B_1 = \{1, x, y, y^2, xy, xy^2\}$, degree 6
- $B_2 = \{1, y\}$, degree 2

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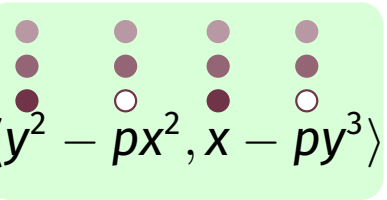

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- Why does x disappear from the staircase?

  Consider $x^4 \cdot x$

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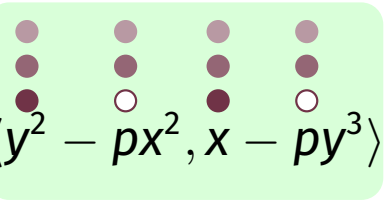

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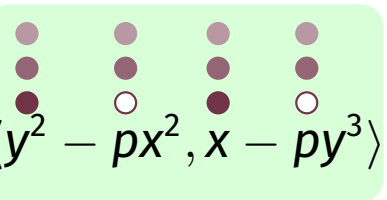

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  Invertible in $K\{x, y\}$

Multiplication matrices and slope factorization

- **Problem**: how to detect this phenomenon in general?

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Multiplication matrix by x

$$T_x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & p^{-1} & 0 & p^{-2} & 0 & p^{-3} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} 1 \\ x \\ y \\ xy \\ y^2 \\ xy^2 \end{matrix}$$

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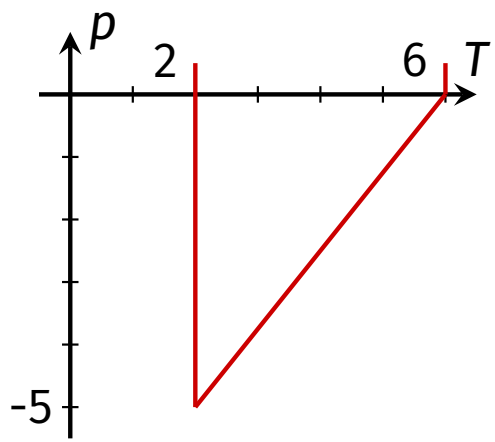
Characteristic poly.

$$T_x^6 - p^{-5} T_x^2 = 0$$



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Slope factorization

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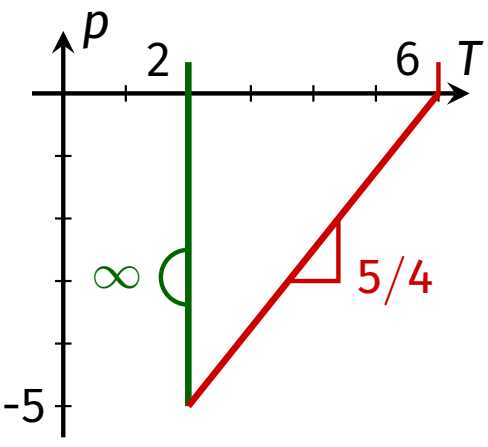

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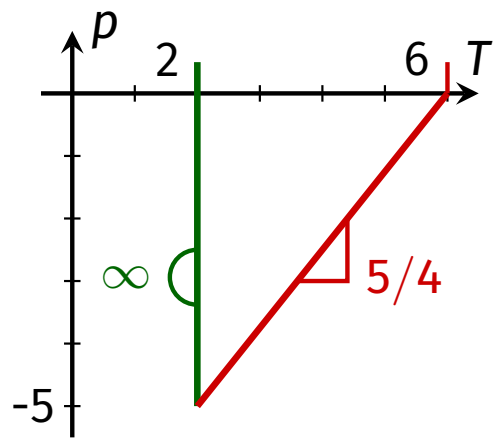


Slope factorization

$$T_x^2 \cdot (T_x^4 - p^{-5}) = 0$$

$\ker(T_x^2)$

Eigenvalues with valuation $\infty \geq 0$

Vectors in the staircase

$\ker(T_x^4 - p^{-5})$

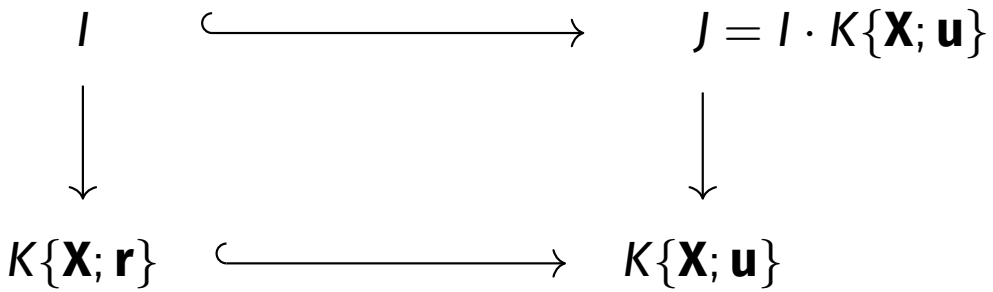
"Eigenvalues" with valuation $-5/4 < 0$

Vectors sent to 0

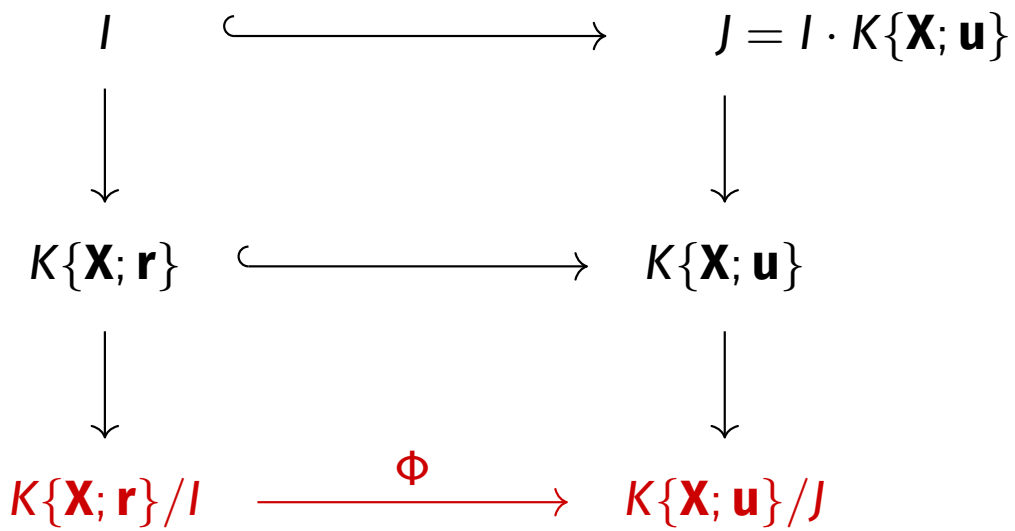
Construction of the new staircase: theoretical point of view

$$K\{\mathbf{X};\mathbf{r}\} \hookrightarrow K\{\mathbf{X};\mathbf{u}\}$$

Construction of the new staircase: theoretical point of view

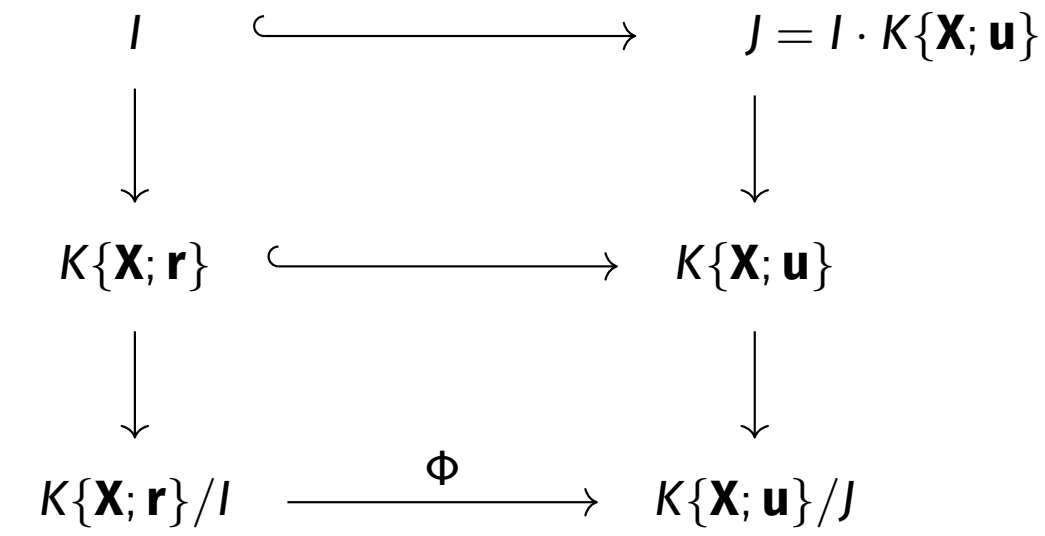


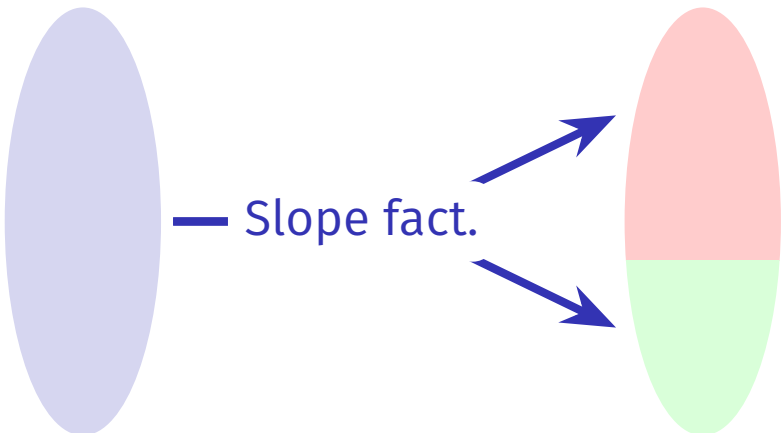
Construction of the new staircase: theoretical point of view



Staircase $B_{\mathbf{r}}$

Construction of the new staircase: theoretical point of view





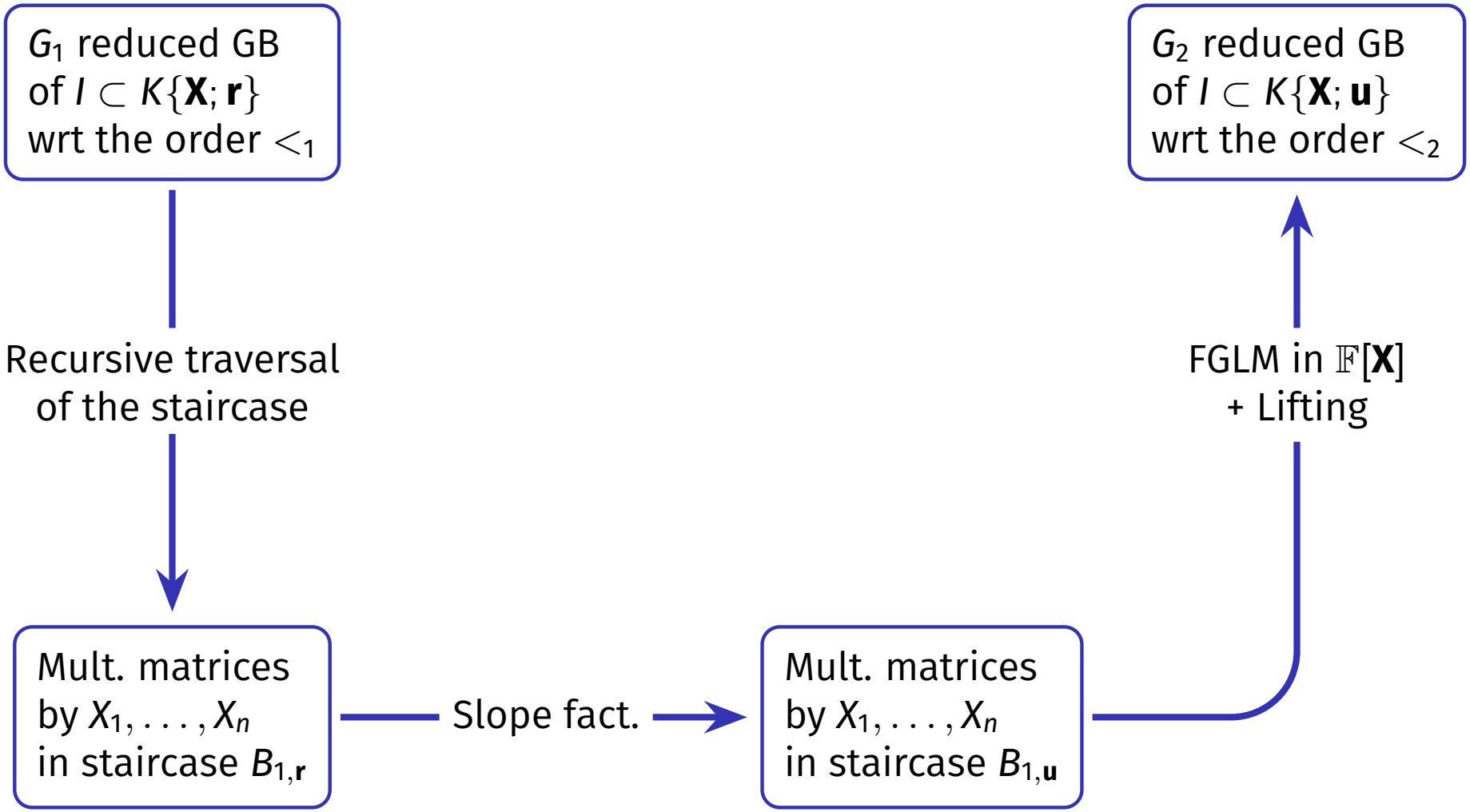
Staircase $B_{\mathbf{r}}$

Basis of $\ker(\Phi)$
$= \sum$ "Eigenspace" of T_i with val. $< u_i$

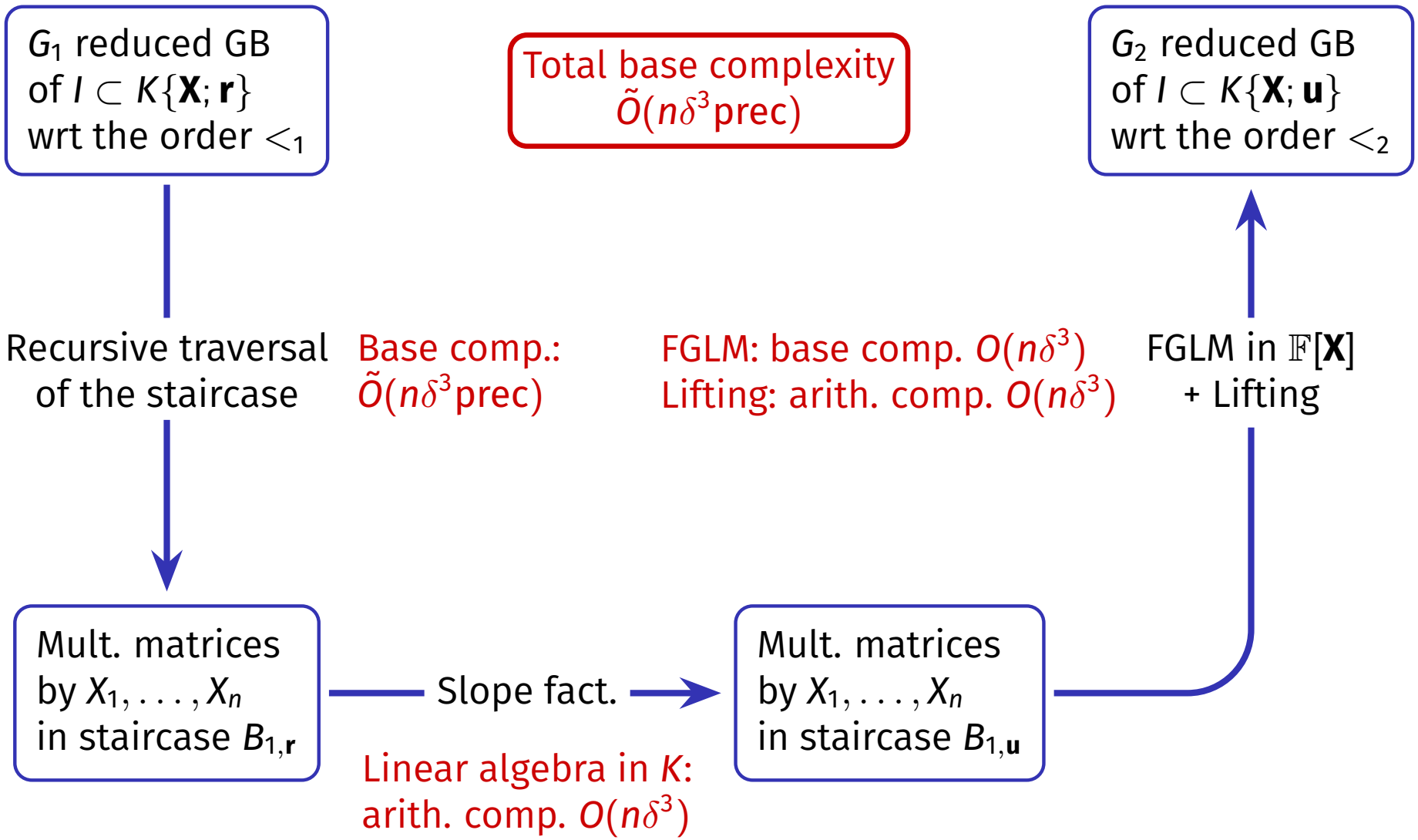
Staircase $B_{\mathbf{u}}$: basis of
$\bigcap$ "Eigenspace" of T_i with val. $\geq u_i$

Topological computation (completion+separation) using linear algebra!

Full FGLM algorithm for Tate algebras



Full FGLM algorithm for Tate algebras



Conclusion

Summary

- FGLM algorithm for Tate series
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- Complexity cubic in degree and quasi-linear in precision

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Thank you for your attention!