FGLM algorithm over Tate algebras

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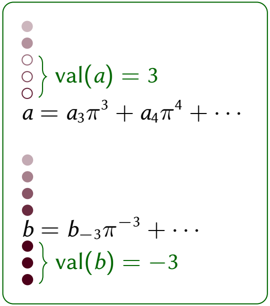
Setting and definitions

Valued field, valuation ring

- Field with a valuation $\mathrm{val} : K \to \mathbb{Z} \cup \infty$ — $\mathbb{Q}_p$ — $k((X))$
- Integer ring $K^\circ = \{x : \mathrm{val}(x) \geq 0\}$ — $\mathbb{Z}_p$ — $k[[X]]$
- Uniformizer π s.t. $\pi K^\circ = \{x : \mathrm{val}(x) \geq 1\}$ — p — X



Metric and topology

- "a is small" $\iff$ "$\mathrm{val}(a)$ is large"
- Non-archimedean metric: "small + small = small"
- $\mathbb{Q}_p$, $\mathbb{Z}_p$, $k((X))$, $k[[X]]$ are complete for that topology

Rigid geometry and Tate series

- "Algebraic geometry, analytic geometry" bridge for non-archimedean geometry
- Main object: Tate series

Tate series

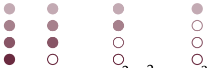
$\mathbf{r} \in \mathbb{Q}^n$: convergence (log)-radii

Definitions

- Tate algebra $K\{X_1, \ldots, X_n; r_1, \ldots, r_n\} = K\{\mathbf{X}; \mathbf{r}\}$
- Set of series $\sum_{\alpha \in \mathbb{N}^n} a_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ with $\mathrm{val}(a_\alpha) - \sum r_j \alpha_j \to \infty$
- *"Convergent for substitutions with* $\mathrm{val}(x_i) \geq -r_i$*"*
- smaller $r_i \iff$ smaller convergence radius $\iff$ larger algebra
- Convention: $r_i = \infty \to$ finitely many terms in X_i (polynomial)

Examples:

- Polynomials are Tate series for all radii (finite sums)



- $f = \sum_{i,j=0}^{\infty} \pi^i X^i = 1 + \pi X + \pi^2 X^2 + \pi^3 X^3 + \cdots$
  - $f \in K\{X\} = K\{X; 0\}$
  - $f \notin K\{X; 1\}$: for all terms, $\mathrm{val}(\pi^\alpha) - \alpha = 0 \not\to \infty$

Gröbner bases over Tate algebras

Gröbner bases:

- Multi-purpose tool for ideal arithmetic in polynomial algebras
- Membership testing, elimination, intersection...
- Uses successive (terminating) reductions
- Requires the definition of an ordering on terms

Construction for Tate series

- Term order considering terms according to the valuation of their coefficient
- First compare $\mathrm{val}(a_\alpha) - \sum r_j \alpha_j$, break ties with a monomial order

$$\cdots > 1\,\mathbf{X}^{\mathbf{i}_1} > \pi\,\mathbf{X}^{\mathbf{i}_2} > \pi \cdot 1 > \pi^2\,\mathbf{X}^{\mathbf{i}_3} > \cdots$$

- Convergent reductions (interrupted at the precision bound) instead of terminating ones
- Allows to use usual algorithms (Buchberger, F4) to compute Gröbner bases

Complexity bottleneck: reductions

Cost of reductions

- Not unusual with Gröbner bases
- Tate case: reductions are interrupted at the precision bound
- The cost grows badly with the precision
- **Question:** can we compute reductions in time quasi-linear in the precision?

Ideas for possible improvement:

- Avoid useless reductions to zero
- Speed-up interreductions
- Exploit overconvergence

Series converging faster, *i.e.*, living in a smaller Tate algebra
Ex: polynomials (log-radii ∞) seen as Tate series

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(Speed-up interreductions, Exploit overconvergence) ⟶ In dim. 0: with FGLM **[this work]**

overconvergence: Series converging faster, *i.e.*, living in a smaller Tate algebra
Ex: polynomials (log-radii ∞) seen as Tate series

FGLM algorithm and applications

Zero-dimensional ideal in $K[\mathbf{X}]$

- Variety = finitely many points
- Quotient $K[\mathbf{X}]/I$ has finite dimension as a vector space over K
- Given a Gröbner basis G, the staircase under G is
  $B = \{m$ monomial not divisible by any LT of $G\}$
- B is a K-basis of $K[\mathbf{X}]/I$
- Key object: matrices of multiplications by $X_1, \ldots, X_n$ in the basis B

Outline of the algorithm:

In: G_1 a reduced Gröbner basis of $I \subset K[\mathbf{X}]$ wrt an order $<_1$

$<_2$ a monomial order

Out: G_2 a reduced Gröbner basis of $I \subset K[\mathbf{X}]$ wrt $<_2$

1. Compute the matrices of multiplication by $X_1, \ldots, X_n$ in the basis B_1 (computing B_1)

2. Convert them into the Gröbner basis G_2 (computing B_2)

FGLM algorithm and applications

Zero-dimensional ideal in $K\{\mathbf{X};\mathbf{r}\}$

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Outline of the algorithm for Tate algebras:

In: G_1 a reduced Gröbner basis of $I \subset K\{\mathbf{X};\mathbf{r}\}$ wrt an order $<_1$
$<_2$ a monomial order, $\mathbf{u} \leq \mathbf{r}$

Out: G_2 a reduced Gröbner basis of $I \cdot K\{\mathbf{X};\mathbf{u}\} \subset K\{\mathbf{X};\mathbf{u}\}$ wrt $<_2$

1. Compute the matrices of multiplication by $X_1, \ldots, X_n$ in the basis $B_{1,\mathbf{r}}$ (computing $B_{1,\mathbf{r}}$)
2. Convert them into matrices in the basis $B_{1,\mathbf{u}}$ (computing $B_{1,\mathbf{u}}$)
3. Convert them into the Gröbner basis G_2 (computing $B_{2,\mathbf{u}}$)

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Outline of the algorithm for Tate algebras:

In: G_1 a Gröbner basis reduced mod π of $I \subset K\{\mathbf{X};\mathbf{r}\}$ wrt an order $<_1$
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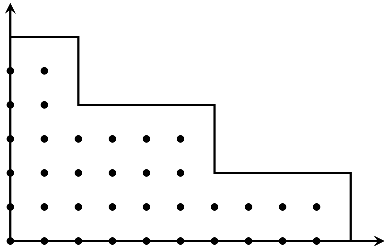
Step 1: computing the multiplication matrices

Idea: we want $\mathrm{NF}(X_i m)$ for all $i \in \{1, \ldots, n\}$, $m \in B$ without computing the NF



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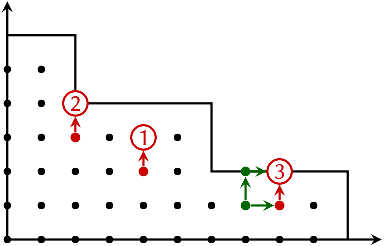

Proceed in increasing order, 3 cases:

1. $X_i m \in B \to \mathrm{NF}(X_i m) = X_i m$
2. $X_i m = \mathrm{LT}(g)$ for $g \in G \to \mathrm{NF}(X_i m) = X_i m - g$
3. Otherwise, write $\mathrm{NF}(X_i m)$ $m = X_j m'$ as linear combination of other normal forms

- Usual case: only involves known normal forms
- Tate case: can involve later monomials, but with coefficients divisible by π
- So we can repeat, increasing the precision by 1 each time

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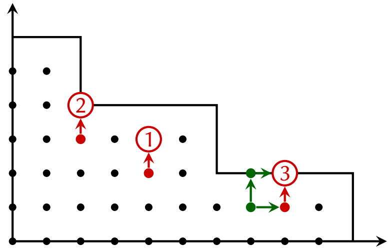

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Improvements

- Recursive algorithm querying the coefficients as needed (instead of relying on the order)
- Fast arithmetic + relaxed algorithms for base complexity quasi-linear in precision
  [van der Hoeven 1997] [Berthomieu, van der Hoeven, Lecerf 2011] [Berthomieu, Lebreton 2012]
- The algorithm only requires that the basis is reduced modulo π → fast interreduction

Step 2: computing the new staircase

Example

- $K[x, y]$: $\mathbf{r} = (\infty, \infty)$
- $\langle \pi x^2 - y^2, \pi y^3 - x \rangle$
- $B_1 = \{1, x, y, y^2, xy, xy^2\}$, degree 6

- $K\{x, y\}$: $\mathbf{u} = (0, 0)$

- $\langle y^2 - \pi x^2, x - \pi y^3 \rangle$
- $B_2 = \{1, y\}$, degree 2!

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- $\langle y^2 - \pi x^2, x - \pi y^3 \rangle$
- $B_1 = \{1, x, y, y^2, xy, xy^2\}$, degree 6
- $B_2 = \{1, y\}$, degree 2!

- Why does x disappear from the staircase?

  Consider $x^4 \cdot x = \frac{1}{\pi} x^3 y^2 = \frac{1}{\pi^2} xy^4 = \frac{1}{\pi^3} x^2 y = \frac{1}{\pi^4} y^3 = \frac{1}{\pi^5} x$

so $x = \pi^5 x^5 = \pi^{10} x^9 = \cdots = 0$ or equivalently $x(1 - \pi^5 x^4) = 0 \implies x = 0$.

Multiplication matrices and slope factorization

- Problem: how to detect this phenomenon in general?

Consider the multiplication matrix by x:

$$T_x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \pi^{-1} & 0 & \pi^{-2} & 0 & \pi^{-3} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} 1 \\ x \\ y \\ xy \\ y^2 \\ xy^2 \end{matrix}$$

$$\begin{matrix} 1 & x & y & xy & y^2 & xy^2 \end{matrix}$$

Characteristic polynomial:

$\chi_x = T^6 - \pi^{-5} T^2$

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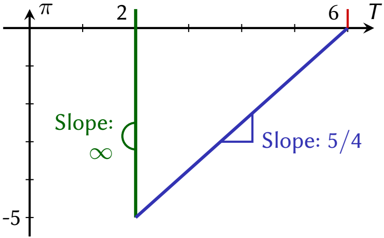
$$\chi_x = T^6 - \pi^{-5} T^2$$
$$= T^2 \cdot (T^4 - \pi^{-5})$$



Slope factorization: split according to the sign of the generalized eigenvalues

- $\ker(T_x^4 - \pi^{-5})$: generalized eigenvectors with valuation $-5/4 < 0$
  $\rightarrow$ vectors sent to 0
- $\ker(T_x^2)$: generalized eigenvectors with valuation $\infty \geq 0$
  $\rightarrow$ vectors in the staircase

Full algorithm and complexity

In: G_1 a reduced Gröbner basis in $K\{\mathbf{X}; \mathbf{r}\}$ wrt an order $<_1$

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$\mathbf{u} \leq \mathbf{r}$ a system of log-radii

Out: G_2 a reduced Gröbner basis wrt $<_2$ in $K\{\mathbf{X}; \mathbf{u}\}$

1. Compute the matrices of multiplication by $X_1, \ldots, X_n$ in the basis $B_{1,\mathbf{r}}$
2. Convert them into matrices of multiplication by $X_1, \ldots, X_n$ in the basis $B_{1,\mathbf{u}}$ (slope factorization)
3. Convert into the basis G_2
   - 3.1 Use the usual algorithm modulo π (in $\mathbb{F}$) to compute $B_{2,\mathbf{u}}$ and $\overline{G_2}$
   - 3.2 Lift the linear algebra operations to obtain G_2

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Complexity

- Step 1 has base complexity $\tilde{O}(n\delta^3\mathrm{prec})$
- Each other step has arithmetic complexity $\tilde{O}(n\delta^3)$
- Arithmetic in K has base complexity quasilinear in precision
- Final base complexity: $\tilde{O}(n\delta^3\mathrm{prec})$

$\delta =$ degree of the ideal $= \#B_{1,\mathbf{r}} \geq \#B_{1,\mathbf{u}}$

Conclusion

Summary

- FGLM algorithm for Tate series
- Allows to perform interreduction and change of log-radii in dimension 0
- Complexity cubic in degree, quasi-linear in precision

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Future work

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- Generalizations of the interreduction in the middle of GB calculations
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Thank you for your attention!

References

- *Gröbner bases over Tate algebras*, ISSAC 2019
- *Signature-based algorithms for Gröbner bases over Tate algebras*, ISSAC 2020
- *On FGLM algorithms with Tate algebras*, preprint 2021