Signature Gröbner bases algorithms over Tate algebras

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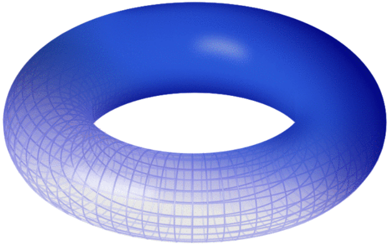
Note: the present slides are images extracted from the presentation video.
For the best viewing experience, please watch the video!

GAGA [Serre, 1956]

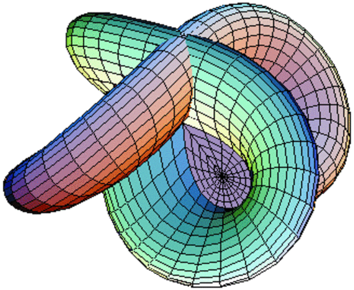
Algebraic Geometry

Multivariate polynomials



Analytic Geometry

Analytic functions



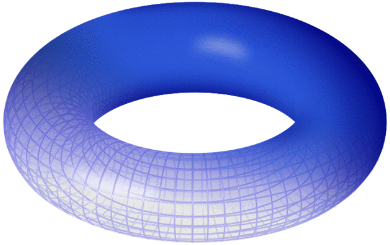
GAGA [Serre, 1956]

Algebraic Geometry

Multivariate polynomials

Analytic Geometry

Analytic functions:
Multivariate power series
convergent on an open ball

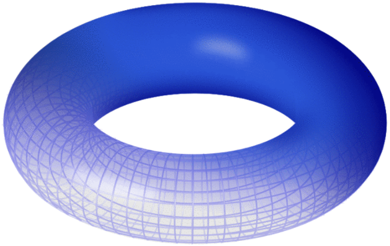


Analytic geometry
in p-adic setting

$\xleftrightarrow{???}$

Algebraic Geometry

Multivariate polynomials



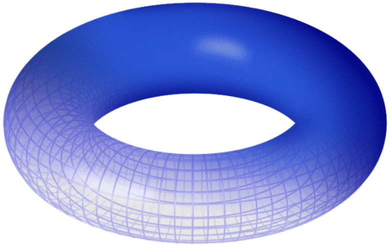
[Tate, 1962]

Rigid Geometry

Tate series:
Multivariate power series
convergent on a closed ball

Algebraic Geometry

Multivariate polynomials



GAGA [Serre, 1956]

Algebraic Geometry $\longleftrightarrow$ Analytic Geometry

Multivariate polynomials

Analytic functions:
Multivariate power series
convergent on an open ball

Effective techniques and software

Exact ideal arithmetic (intersection, union...):

Gröbner bases...

Rigid Geometry

Tate series:
Multivariate power series
convergent on a closed ball

[Tate, 1962]

Algebraic Geometry

Multivariate polynomials

Our work:

- Theory of Gröbner bases
- Buchberger's algorithm
- Signature-based algorithms

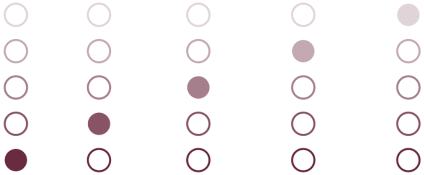
Effective techniques and software

Exact ideal arithmetic (intersection, uni

Gröbner bases...

Tate series: Multivariate power series with coefficients in a valued ring, convergent on a closed ball

Valued ring: $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{C}[[X]]$, $\mathbb{C}((X))$...



$$a = a_0 + a_1\pi + a_2\pi^2 + a_3\pi^3 + a_4\pi^4 + \cdots$$

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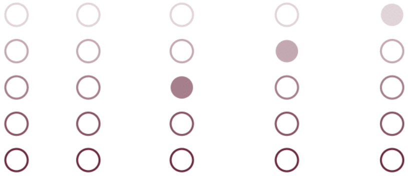


$$a = \underbrace{0 + 0\pi}_{\text{Valuation}=2} + a_2\pi^2 + a_3\pi^3 + a_4\pi^4 + \cdots$$

Tate series: Multivariate power series with coefficients in a valued ring, convergent on a closed ball

Valued ring: $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{C}[[X]]$, $\mathbb{C}((X))$…

Valuation=2

$a \in K^o$

Tate series: Multivariate power series with coefficients in a valued ring, convergent on a closed ball

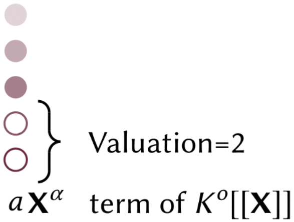
Valued ring: $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{C}[[X]]$, $\mathbb{C}((X))\ldots$

Valuation=2

$a\,X_1^{\alpha_1}\cdots X_n^{\alpha_n}$ term of $K^o[[\mathbf{X}]]$

Tate series: Multivariate power series with coefficients in a valued ring, convergent on a closed ball

Valued ring: $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{C}[[X]]$, $\mathbb{C}((X))\dots$



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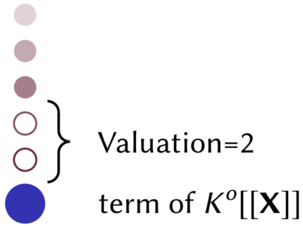

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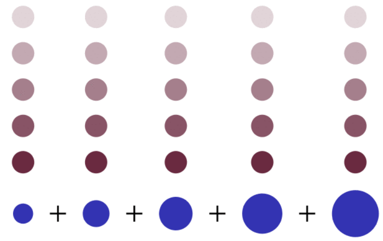

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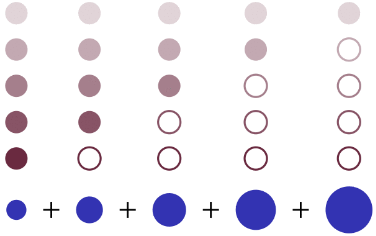
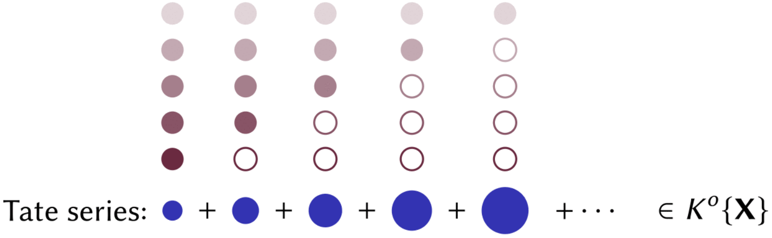


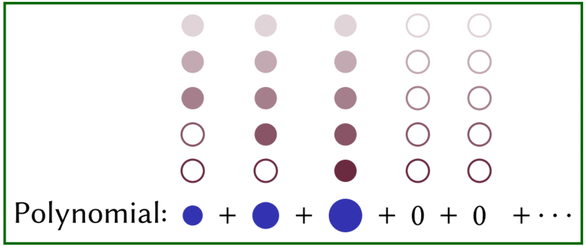
Tate series: $+ \cdots \quad \in K^{o}\{\mathbf{X}\}$

Convergence condition: the valuation goes to infinity

Tate series: Multivariate power series with coefficients in a valued ring, convergent on a closed ball

Valued ring: $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{C}[[X]]$, $\mathbb{C}((X))$...





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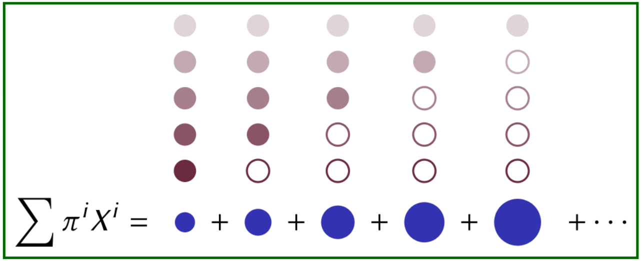

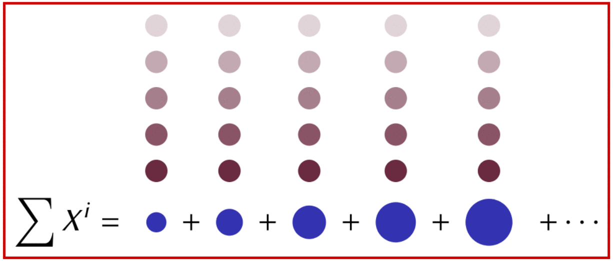

Gröbner bases in finite precision:

- Need to work around error propagation
- Need to perform tests to zero to find leading terms

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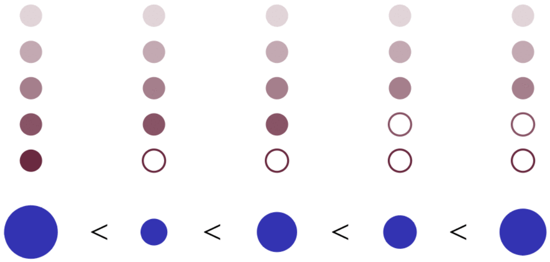
Solution: term ordering such that "close to zero" means small

- Order the terms with their coefficients
- First compare the valuations
- Then break ties with a monomial order



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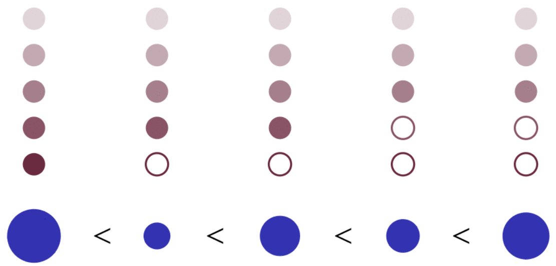

Properties:

- Terms with smaller valuation are larger
- Infinite reductions have increasing valuation
- Tate series have a leading term

Buchberger's algorithm

Input: F list of Tate series

Output: G Gröbner basis of the ideal $\langle F \rangle$

1. $G \leftarrow F$
2. $\mathcal{P} \leftarrow \{\text{S-pol}(g, g') : g \neq g' \in G\}$
3. While $\mathcal{P} \neq \emptyset$:
4. $\quad h \leftarrow$ an element of $\mathcal{P}$
5. $\quad h \leftarrow \text{Reduce}(h, G)$
6. $\quad$ If $h \neq 0$:
7. $\quad\quad$ Add h to G
8. $\quad\quad$ Add to $\mathcal{P}$ all $\text{S-Pol}(g, h)$ for $g \in G$
9. Return G

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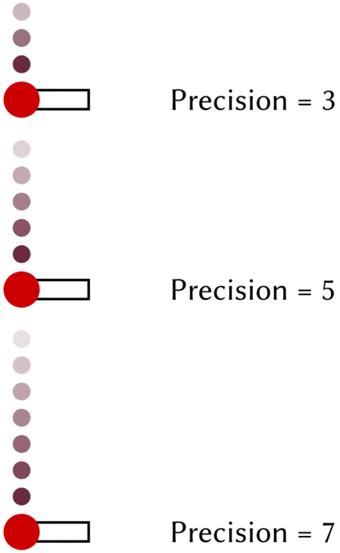

Bottleneck: reductions to zero

Increasing the precision makes it worse!

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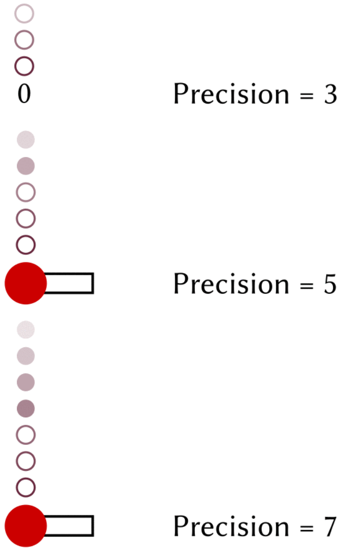

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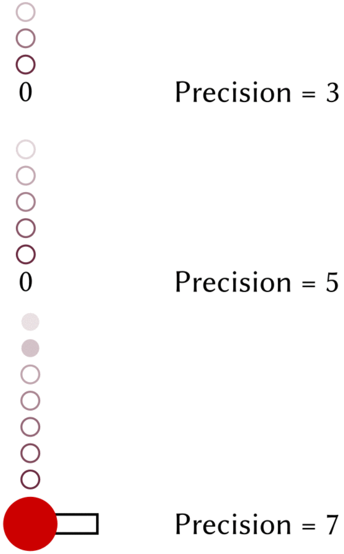

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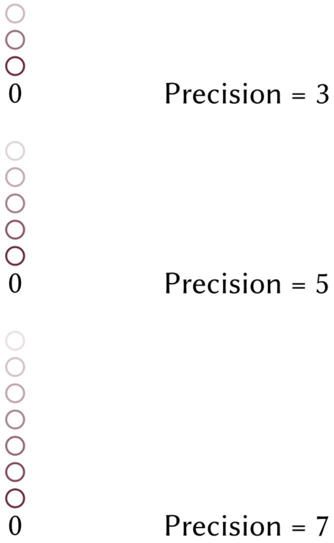

Bottleneck: reductions to zero

Increasing the precision makes it worse!

Some reductions to zero are predictable, how to detect them?

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$$
\begin{array}{rl}
p & \in \mathbb{C}[\mathbf{X}] \\
+ & \\
q & \in \mathbb{C}[\mathbf{X}] \\
\shortparallel & \\
p + q & \in \mathbb{C}[\mathbf{X}] \\
{\color{red}0} &
\end{array}
$$

Some reductions to zero are predictable, how to detect them?

$$
\begin{array}{ccccccccccc}
p & = & p_1 & f_1 + & p_2 & f_2 + \cdots + & p_m & f_m \in \mathbb{C}[\mathbf{X}] \\
+ & & & & & & & \\
q & = & q_1 & f_1 + & q_2 & f_2 + \cdots + & q_m & f_m \in \mathbb{C}[\mathbf{X}] \\
\shortparallel & & & & & & & \\
p + q & = & (p_1 + q_1) & f_1 + & (p_2 + q_2) & f_2 + \cdots + & (p_m + q_m) & f_m \in \mathbb{C}[\mathbf{X}] \\
0 & & f_2 & & -f_1 & & 0 & f_2 f_1 - f_1 f_2 = 0
\end{array}
$$

Some reductions to zero are predictable, how to detect them?

Idea 1: keep track of the module representation of the elements [Möller, Mora, Traverso, 1992]

$$
\begin{array}{ccccccccc}
p & & p_1 \mathbf{e}_1 & + & p_2 \mathbf{e}_2 & + \cdots + & p_m \mathbf{e}_m & \in \mathbb{C}[\mathbf{X}]^m \\
+ & & & & & & & \\
q & & q_1 \mathbf{e}_1 & + & q_2 \mathbf{e}_2 & + \cdots + & q_m \mathbf{e}_m & \in \mathbb{C}[\mathbf{X}]^m \\
\parallel & & & & & & & \\
p+q & & (p_1+q_1)\mathbf{e}_1 & + & (p_2+q_2)\mathbf{e}_2 & + \cdots + & (p_m+q_m)\mathbf{e}_m & \in \mathbb{C}[\mathbf{X}]^m \\
0 & & f_2 & & -f_1 & & 0 & f_2\mathbf{e}_1 - f_1\mathbf{e}_2 : \text{known syzygy}
\end{array}
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+ & & & & & & & & \\
q & & q_1 \mathbf{e}_1 & + & q_2 \mathbf{e}_2 & + \cdots + & q_m \mathbf{e}_m & \in \mathbb{C}[\mathbf{X}]^m & \\
\shortparallel & & & & & & & & \\
p+q & & (p_1+q_1)\mathbf{e}_1 & + & (p_2+q_2)\mathbf{e}_2 & + \cdots + & (p_m+q_m)\mathbf{e}_m & \in \mathbb{C}[\mathbf{X}]^m & \\
0 & & f_2 & & -f_1 & & 0 & & f_2\mathbf{e}_1 - f_1\mathbf{e}_2 : \text{known syzygy} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & &
\end{array}
$$

Cost: $m+1$ polynomial additions

Some reductions to zero are predictable, how to detect them?

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\begin{array}{ccccccccc}
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+ & & & & & & & \\
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\| & & & & & & & \\
p + q & = & (p_1 + q_1) & f_1 + & (p_2 + q_2) & f_2 + \cdots + & (p_m + q_m) & f_m \in \mathbb{C}[\mathbf{X}] \\
0 & & f_2 & & -f_1 & & 0 & f_2 f_1 - f_1 f_2 = 0
\end{array}
$$

Cost: 1 polynomial addition

Some reductions to zero are predictable, how to detect them?

Idea 1: keep track of the module representation of the elements [Möller, Mora, Traverso, 1992]

$$
\begin{array}{ccc}
p & p_1 \mathbf{e}_1 + p_2 \mathbf{e}_2 + \cdots + p_m \mathbf{e}_m & \in \mathbb{C}[\mathbf{X}]^m \\
+ & & \\
q & q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + \cdots + q_m \mathbf{e}_m & \in \mathbb{C}[\mathbf{X}]^m \\
\shortparallel & & \\
p+q & (p_1+q_1)\mathbf{e}_1 + (p_2+q_2)\mathbf{e}_2 + \cdots + (p_m+q_m)\mathbf{e}_m & \in \mathbb{C}[\mathbf{X}]^m \\
0 & f_2 \qquad -f_1 \qquad\qquad 0 & f_2\mathbf{e}_1 - f_1\mathbf{e}_2 : \text{known syzygy}
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Cost: $m+1$ polynomial additions

Some reductions to zero are predictable, how to detect them?

Idea 1: keep track of the module representation of the elements [Möller, Mora, Traverso, 1992]

Idea 2: only keep some terms of the module elements

$$
\begin{array}{ccl}
p & & \mathrm{LT}(p_1)\,\mathbf{e}_1 + \mathrm{LT}(p_2)\,\mathbf{e}_2 + \cdots + \mathrm{LT}(p_m)\,\mathbf{e}_m + \text{ smaller terms} \in \mathbb{C}[\mathbf{X}]^m \\
+ & & \\
q & & \mathrm{LT}(q_1)\,\mathbf{e}_1 + \mathrm{LT}(q_2)\,\mathbf{e}_2 + \cdots + \mathrm{LT}(q_m)\,\mathbf{e}_m + \text{ smaller terms} \in \mathbb{C}[\mathbf{X}]^m \\
\shortparallel & & \\
p+q & & \mathrm{LT}(p_1+q_1)\mathbf{e}_1 + \mathrm{LT}(p_2+q_2)\mathbf{e}_2 + \cdots + \mathrm{LT}(p_m+q_m)\mathbf{e}_m + \text{ smaller terms} \in \mathbb{C}[\mathbf{X}]^m \\
0 & & \mathrm{LT}(f_2) \qquad -\mathrm{LT}(f_1) \qquad\qquad 0 \qquad \mathrm{LT}(f_2)\mathbf{e}_1 - \mathrm{LT}(f_1)\mathbf{e}_2 :
\end{array}
$$

LT of a known syzygy

Cost: 1 polynomial addition and m term comparisons : $\mathrm{LT}(p+q) = \begin{cases} \mathrm{LT}(p) & \text{if } \mathrm{LT}(p) > \mathrm{LT}(q) \\ \mathrm{LT}(q) & \text{if } \mathrm{LT}(p) < \mathrm{LT}(q) \\ ??? & \text{otherwise} \end{cases}$

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\begin{array}{ccl}
p & \mathrm{LT}(p_i) & \mathbf{e}_i + \text{ smaller terms} \in \mathbb{C}[\mathbf{X}]^m \\
+ & & \\
q & \mathrm{LT}(q_i) & \mathbf{e}_i + \text{ smaller terms} \in \mathbb{C}[\mathbf{X}]^m \\
\parallel & & \\
p+q & \mathrm{LT}(p_i+q_i) & \mathbf{e}_i + \text{ smaller terms} \in \mathbb{C}[\mathbf{X}]^m \\
0 & \mathrm{LT}(f_2) & \qquad \mathrm{LT}(f_2)\mathbf{e}_1 :
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Some reductions to zero are predictable, how to detect them?

Idea 1: keep track of the module representation of the elements [Möller, Mora, Traverso, 1992]

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Idea 3: skip operations which cannot be done [Faugère, 2002]

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\begin{array}{ccl}
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0 & \mathrm{LT}(f_2) & \qquad \mathrm{LT}(f_2)\mathbf{e}_1 : \\
\uparrow & \uparrow & \qquad \text{LT of a known syzygy}
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- Consider pairs instead of polynomials:

$$(s, p)$$

$$p = \sum p_i f_i \in \mathbb{C}[\mathbf{X}]$$

Signature: $s = \mathrm{LT}(\sum p_i \mathbf{e}_i)$ term of $\mathbb{C}[\mathbf{X}]^m$

- Only allow regular operations:

$$(s, f) + (t, g) = \begin{cases} (s, f+g) & \text{if } s > t \\ (t, f+g) & \text{if } s < t \\ \text{non-regular otherwise} \end{cases}$$

Some reductions to zero are predictable, how to detect them?

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- Consider pairs instead of polynomials: Survey: [Eder, Faugère, 2017]

$$(s, p)$$

$$\longrightarrow p = \sum p_i f_i \in \mathbb{C}[\mathbf{X}]$$

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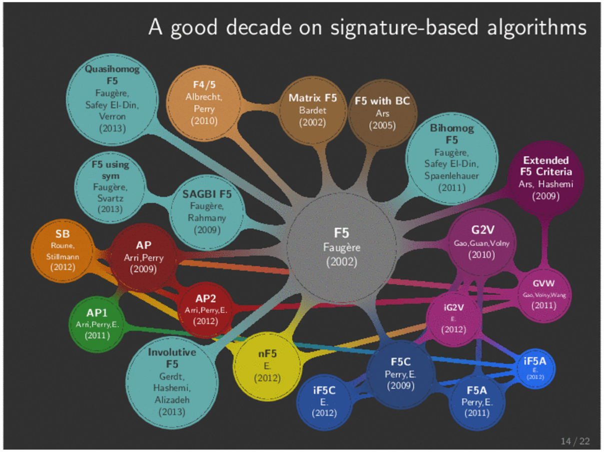

Image: Christian Eder, 2013

Buchberger's algorithm

Input: F list of Tate series

Output: G Gröbner basis of the ideal $\langle F \rangle$

1. $G \leftarrow F$
2. $\mathcal{P} \leftarrow \{\text{S-pol}(g, g') : g \neq g' \in G\}$
3. While $\mathcal{P} \neq \emptyset$:
4. $\quad h \leftarrow$ an element of $\mathcal{P}$
5. $\quad h \leftarrow \text{Reduce}(h, G)$
6. $\quad$ If $h \neq 0$:
7. $\quad\quad$ Add h to G
8. $\quad\quad$ Add to $\mathcal{P}$ all $\text{S-Pol}(g, h)$ for $g \in G$
9. Return G

Signature-based algorithm

Input: $F = \{f_1, \dots, f_n\}$ list of polynomials

Output: G Gröbner basis of the ideal $\langle F \rangle$

1. $G \leftarrow \{(\mathbf{e}_i, f_i) : f_i \in F\}$
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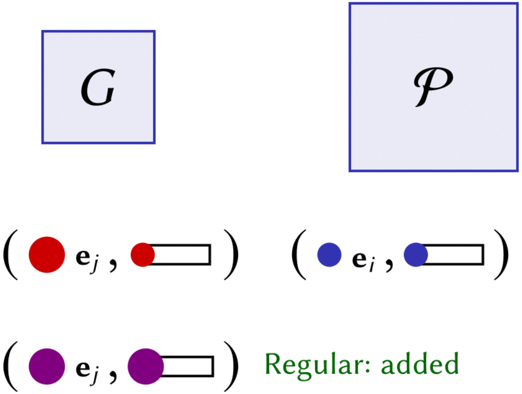

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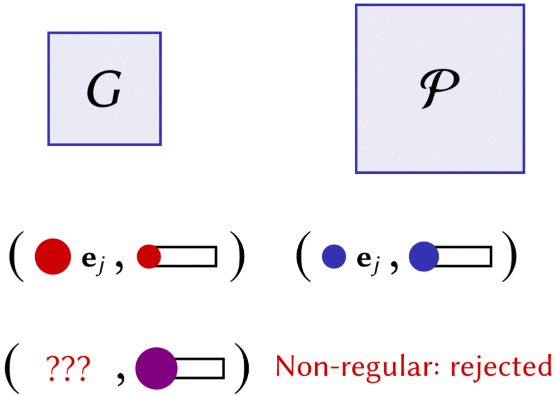

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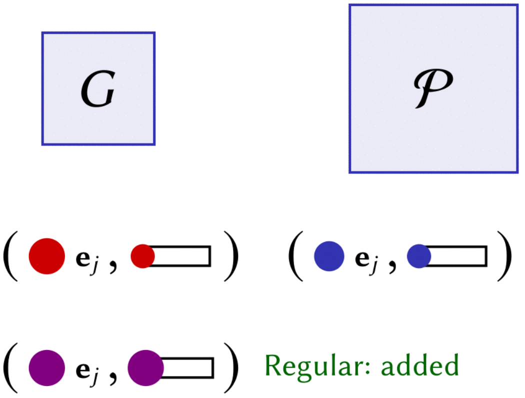

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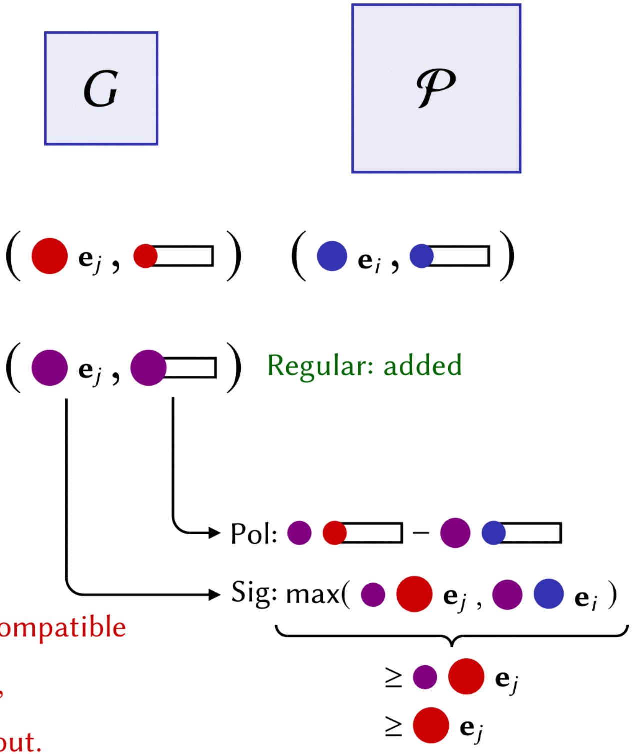

If the signature ordering is compatible

with the monomial ordering,

signatures increase throughout.

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Choice of a signature ordering:

- Compatible with the monomial ordering
- Gives a loop invariant for the algorithm

Easiest example: PoTe: Position over Term

$$\bullet\,\mathbf{e}_1 < \bullet\,\mathbf{e}_2 < \bullet\,\mathbf{e}_2 < \bullet\,\mathbf{e}_3 < \bullet\,\mathbf{e}_3$$

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Current signature: $\bullet\, \mathbf{e}_i$

Current polynomial: $\bullet\, f_i$ + sum of previous

Current basis: basis of $\{f_1, \dots, f_{i-1}\}$

Incremental signature-based algorithm [G2V, 2010]

Input: $F = \{f_1, \dots, f_n\}$ list of polynomials

Output: G Gröbner basis of the ideal $\langle F \rangle$

1. $G \leftarrow \emptyset$
2. For i from 1 to n:
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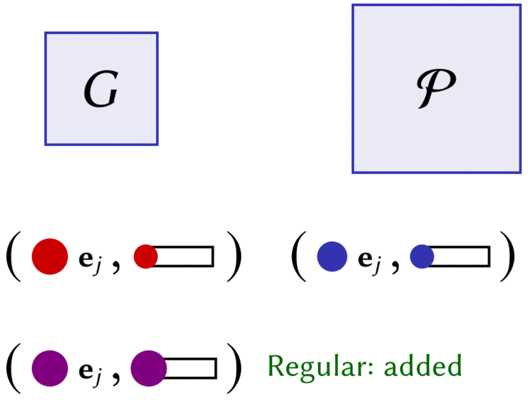

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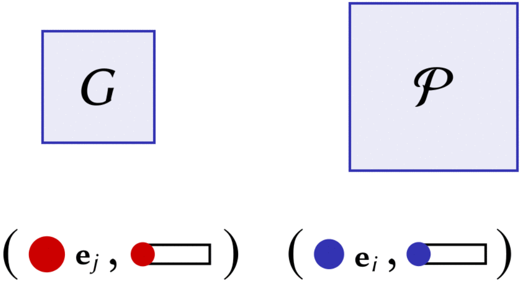

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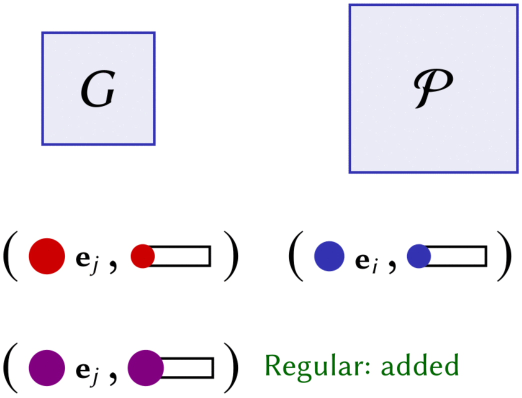

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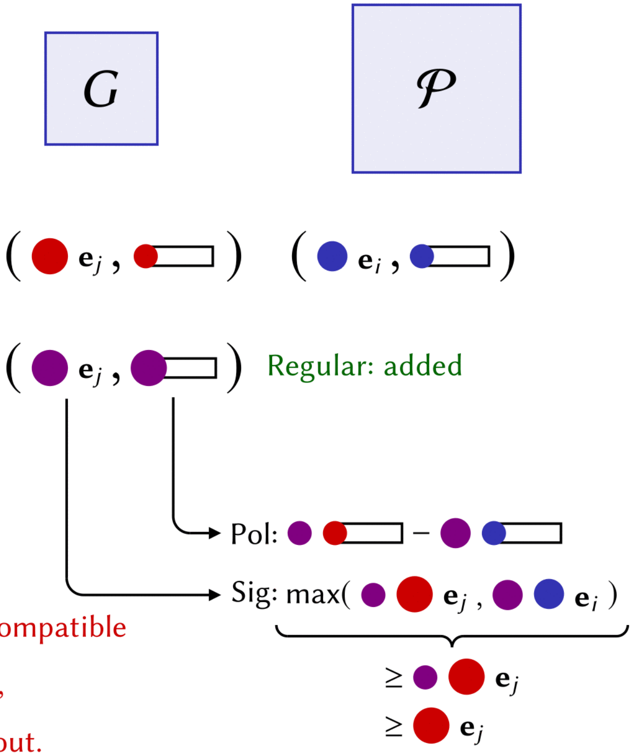

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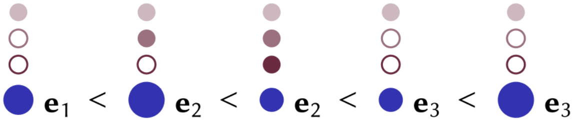

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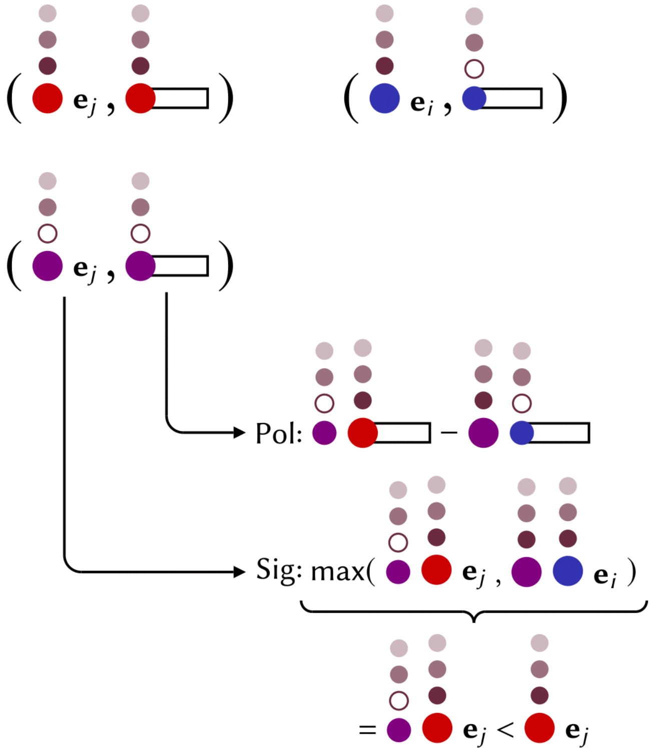
But signatures can decrease!



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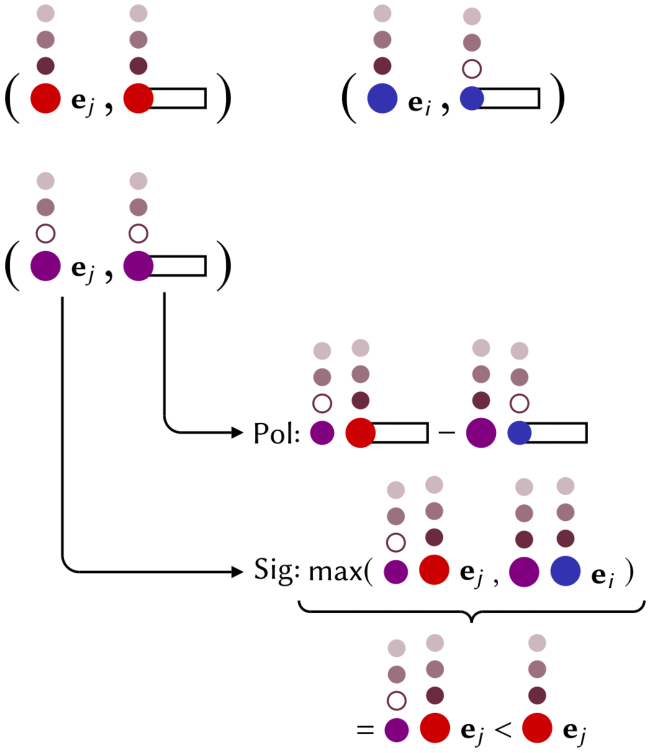

What is the difference with the polynomial case?

- The polynomial monomial order is global : $1 \leq$
- Most signature-based algorithms require a global order
- There are also local orders : ≤ 1
- and signature-based algorithms for that case

  [Lu et.al. 2018]

- Our order is mixed : $< 1 <$
- The local proof can be adapted

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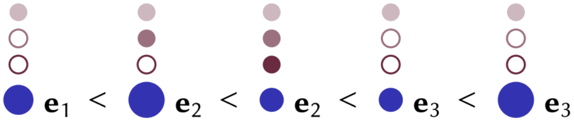
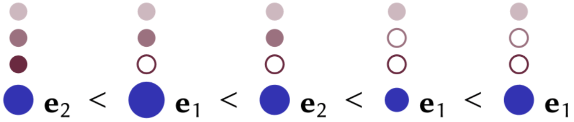
Signature ordering 1: Position over Term



Signature ordering 2: VaPoTe

Increasing Valuation over Position over Term



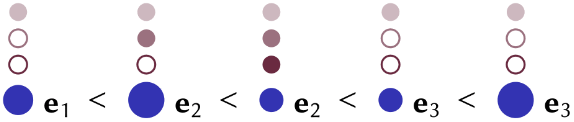
Incremental signature-based algorithm "VaPoTe"

Input: $F = \{f_1, \dots, f_n\}$ list of Tate series

Output: G Gröbner basis of the ideal $\langle F \rangle$

1. (Initialization), $Q \leftarrow \{(\mathbf{e}_i, f_i)\}$
2. For v from 0 to ∞:
3. For each element with valuation v in Q
4. (Update the basis like in PoTe)
5. $\dots$
6. If $\mathrm{val}(h) > v$:
7. Add h to Q
8. Else:
9. Update $\mathcal{P}$
10. Return G

Signature ordering 1: Position over Term



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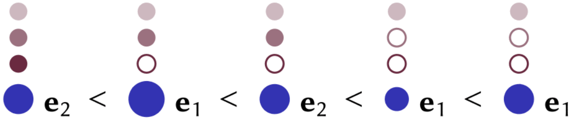
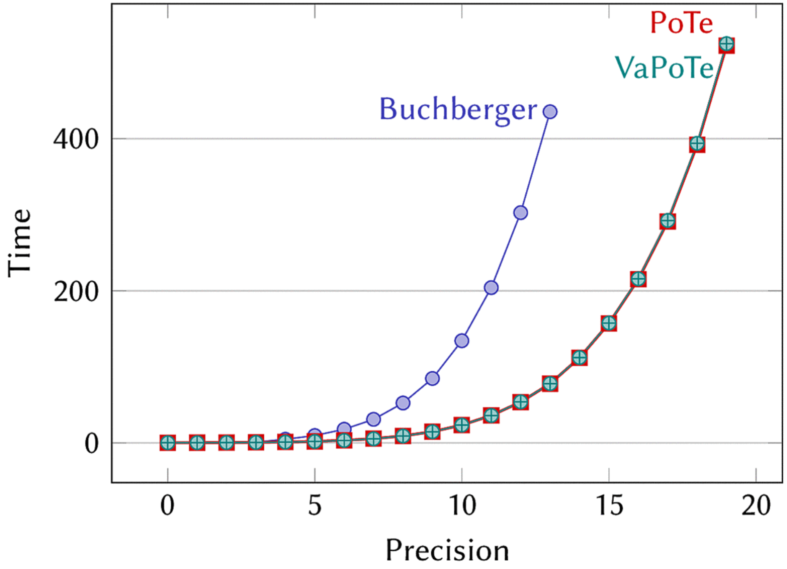


Conclusion

- Two algorithms: PoTe and VaPote
- Incremental, signature-based
- Generically equivalent
- Usually faster than Buchberger
- Distributed with SageMath 9.1



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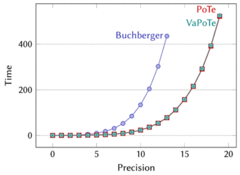

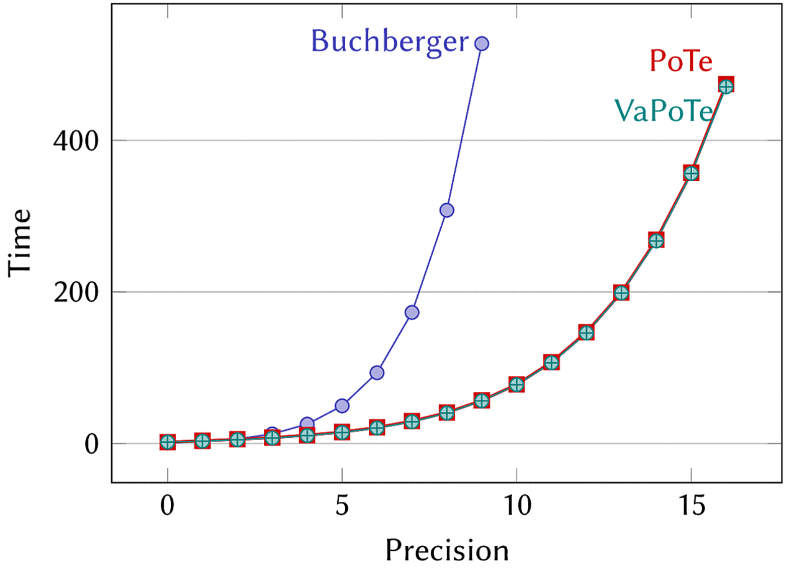

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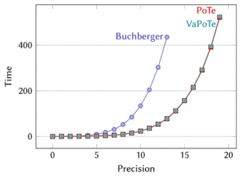
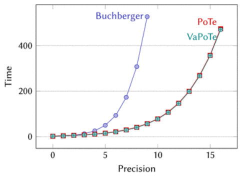
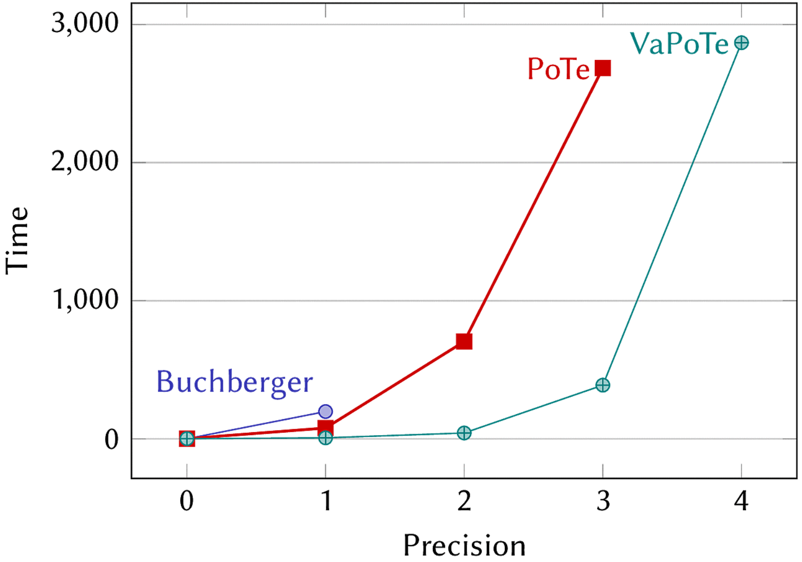




Future work

- Understand non-generic performance differences



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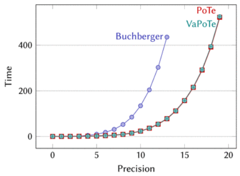
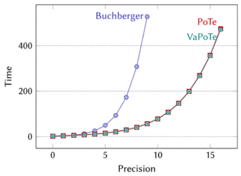
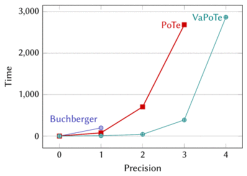
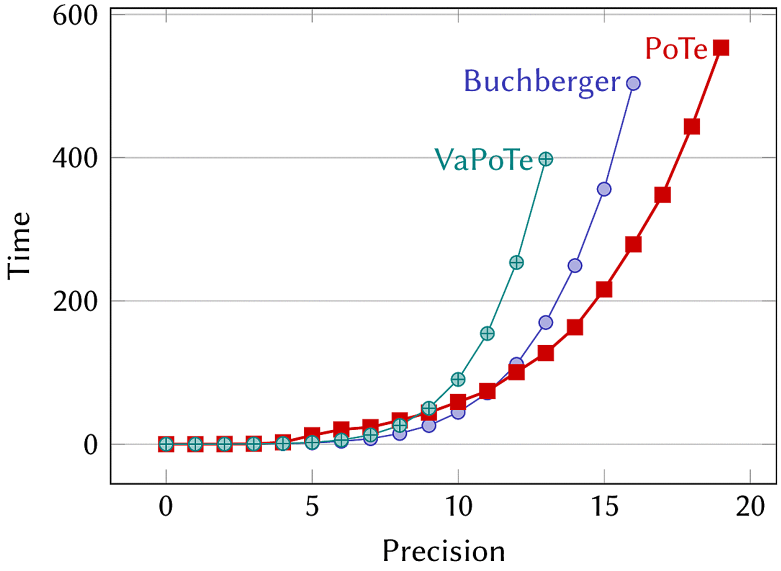






Future work

- Understand non-generic performance differences



Conclusion

- Two algorithms: PoTe and VaPote
- Incremental, signature-based
- Generically equivalent
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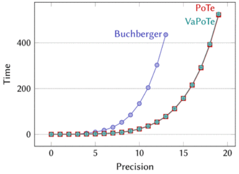

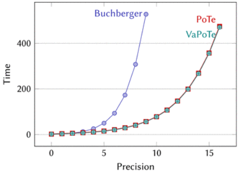

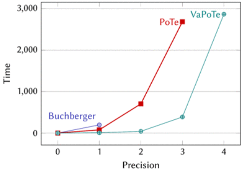

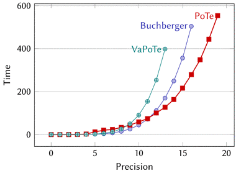

Future work

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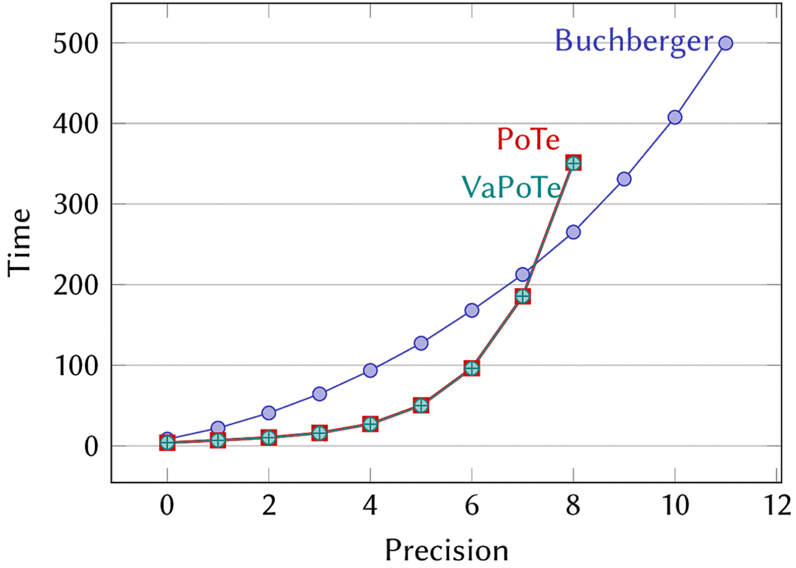

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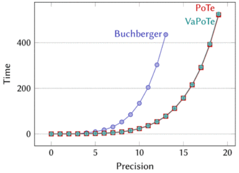

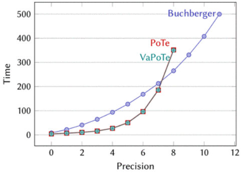

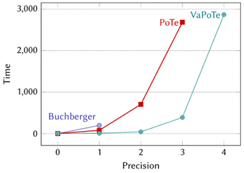

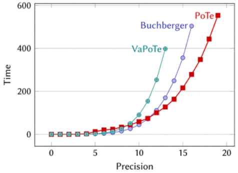

Future work

- Understand non-generic performance differences
- Examine possible optimizations between the loops
- Flatten the curve in precision