Computer algebra algorithms for solving polynomial systems, software and applications

Thibaut Verron

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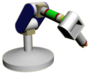
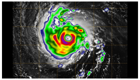
Non-linear modelization and computer algebra





Applications → Robotics | Cryptography | Dynamical systems | …

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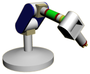
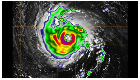
System of polynomial equations (and inequations)

↓

Solutions





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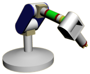
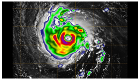
Many tools:

- Numeric
  - Ex: Newton iteration, homotopy...
  - Trade precision for speed
- Computer Algebra
  - Ex: Resultants, Gröbner bases...
  - Exact, exhaustive and certifiable
- Hybrid

Non-linear modelization and computer algebra





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Generic and structured systems

Goal: exact, exhaustive and certified results

- Replace or supplement numeric calculations with symbolic manipulations
- Difficulty: intrinsic complexity of the objects being computed

Examples:

- NP-complete problem over finite fields
- Bézout bound: number of solutions exponential (product of the degrees)
- Worst case: doubly exponential space complexity [Mayr, Meyer 1984]
- For generic system, singly exponential bounds (time and space)

In practice, systems from applications are...

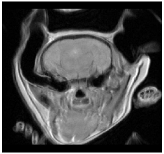
- ... not generic
- ... not instances of the worst case complexity

Key question: identify underlying structures to recover the generic complexity

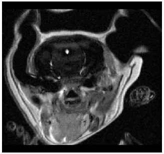
An example: algebraic classification for magnetic resonance imagery

(With B. Bonnard, J.-C. Faugère, A. Jacquemard and M. Safey El Din)

- Context: Magnetic Resonance Imagery
- Goal: optimize contrast



Bad contrast



Optimized

- Optimal control approach: the Bloch model

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} y_i = -\Gamma_i\, y_i - u\, z_i \\ \dfrac{\mathrm{d}}{\mathrm{d}t} z_i = -\gamma_i\, (1 - z_i) + u\, y_i \end{cases} \qquad (i = 1, 2, \ldots, n)$$

y_i, z_i : $2n$ dynamic variables

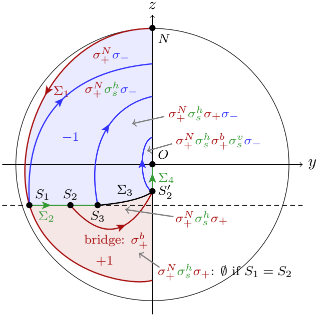
Bloch ball: $y_i^2 + z_i^2 \leq 1$

u : control function

γ_i, Γ_i : $2n$ physical parameters fixed by the experimental setting

$\gamma_i > 0, \Gamma_i > 0, 2\,\Gamma_i \geq \gamma_i$

Semi-algebraic classification problem for MRI



Problem: classification of optimal trajectories

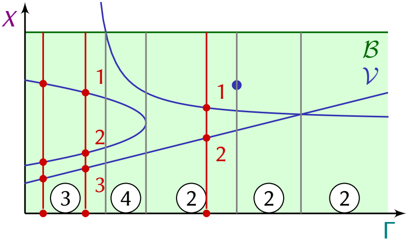
- Control of a single particle: done
- For two particles: more complicated
- Classify some algebraic invariants instead
- Used for choosing simulations to run

Example of algebraic invariant:

- Linked to equilibrium points
- Equations:

$$\mathcal{V} = \left\{ D = \frac{\partial D}{\partial y_1} = \frac{\partial D}{\partial z_1} = \frac{\partial D}{\partial y_2} = \frac{\partial D}{\partial z_2} = 0 \right\}$$

- D : determinant of 4 vector fields
- Inequalities: $\mathcal{B} = \{ y_i^2 + z_i^2 \leq 1 \}$
- Classification question: real points of $\mathcal{V} \cap \mathcal{B}$ depending on γ_i, Γ_i



Results for the MRI classification problem

State of the art:

- Existing tools can't solve the problem efficiently
- 1000s on the case of water (easier: $\gamma_1 = \Gamma_1 = 1$), full problem out of reach
- Complicated output for further steps

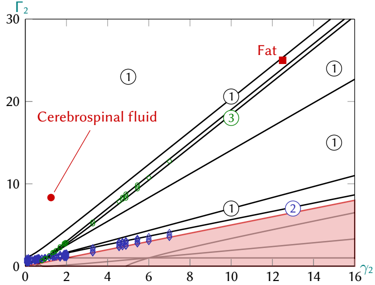
Results:

- Dedicated algorithm exploiting the structure of the system (determinants of matrices)
- Implemented in Maple
- Used to give full classification to the application
- 10s on the case of water, 4h on the full problem

Tools:

- Real geometry: Whitney stratification, Thom's isotopy theorem, critical points
- Algebra: determinantal ideals, incidence varieties
- Computer Algebra: polynomial elimination

Classification in the case of water ($\gamma_1 = \Gamma_1 = 1$)



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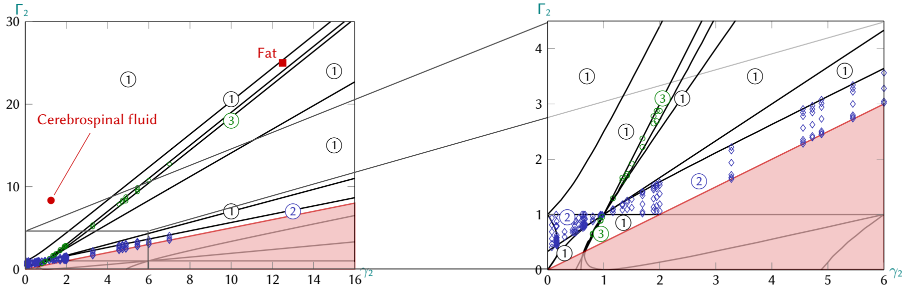

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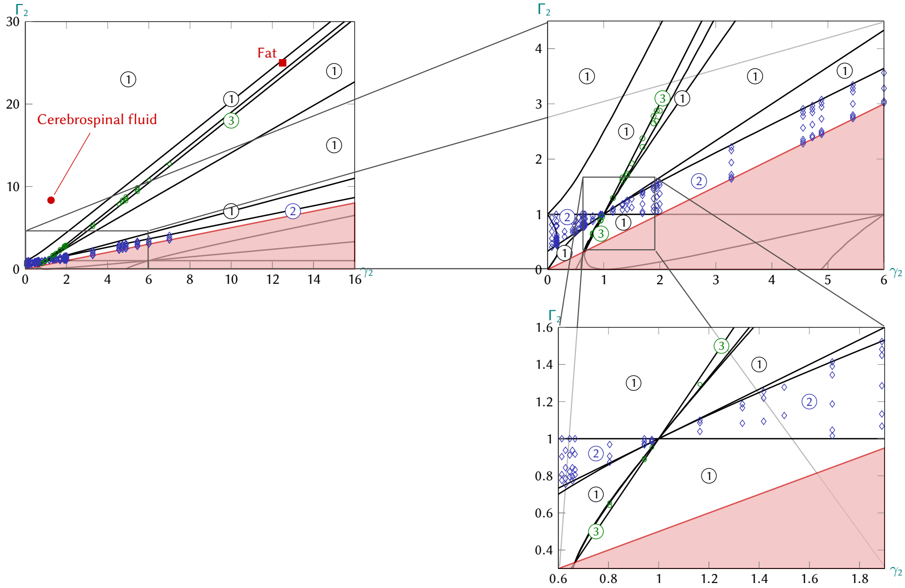

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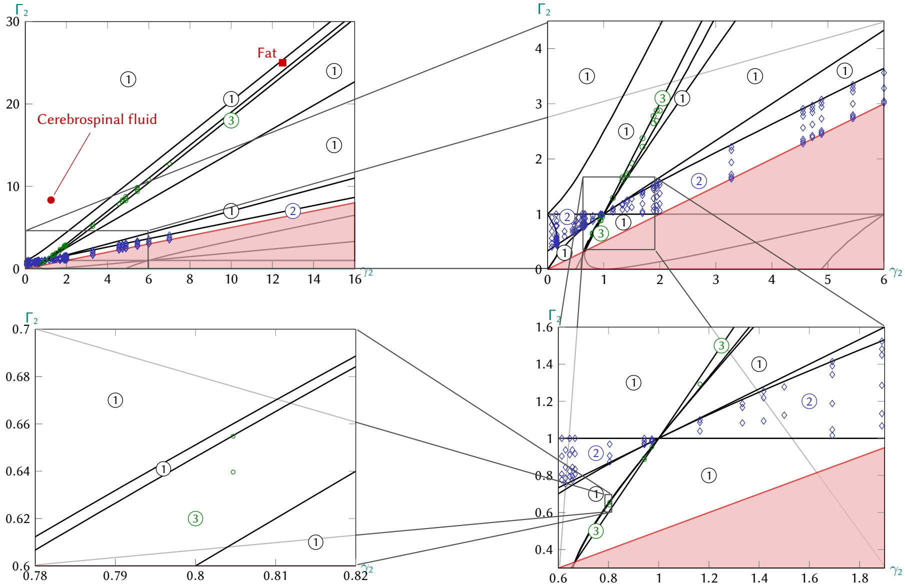

Main computer algebra building block : polynomial elimination

Polynomial elimination:

- Given an ideal $I \subset K[X_1, \ldots, X_n, G_1, \ldots, G_r]$
- Compute a basis of $I_G = I \cap K[G_1, \ldots, G_r]$

Computing eliminations allows to...

- ... compute projections of varieties
- ... solve if finitely many solutions (by iterating)
- ... compute unions and differences of varieties (by lifting)

Many tools: resultants, triangular sets, Gröbner bases

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$$\left\{\begin{array}{l} 2{X_1}^2 G_1 - 3{X_2}^2 G_2 - 3{G_2}^2 \\ X_1 G_1 + 2X_2 G_2 \\ X_1 X_2 + 4G_1 G_2 - 8{G_2}^2 \end{array}\right. \longrightarrow \left\{\begin{array}{l} X_1 X_2 + 4G_1 G_2 - 8{G_2}^2 \\ X_1 G_1 + 2X_2 G_2 \\ 32X_1 {G_2}^3 + 3X_1 {G_2}^2 - 12X_2 G_1 {G_2}^2 + 56X_2 {G_2}^3 \\ 3{X_2}^2 G_2 - 16G_1 {G_2}^2 + 32{G_2}^3 + 3{G_2}^2 \\ 6{G_1}^2 G_2 - 28G_1 {G_2}^2 + 32{G_2}^3 + 3{G_2}^2 \end{array}\right.$$

Other previous works

Structured

Polynomial systems

MRI, control

Determinantal
Critical points

Real geometry

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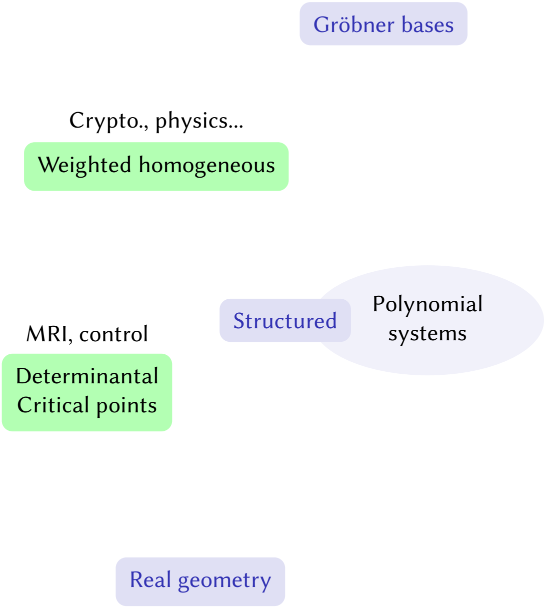

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GB over rings
Signature GB

Number theory

Gröbner bases on $\mathbb{Z}$

Crypto., physics...

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Tool for *e.g.* cryptography: weighted homogeneous systems

(With J.C. Faugère and M. Safey El Din)

Example: system from the discrete logarithm problem [Faugère, Gaudry, Huot, Renault, 2013]

$$0 = \begin{bmatrix}41518\\33900\\8840\\22855\\29081\end{bmatrix} X_5^{16} + \begin{bmatrix}49874\\32136\\34252\\24932\\11782\end{bmatrix} X_1^{8} + \begin{bmatrix}45709\\10698\\45336\\26076\\55993\end{bmatrix} X_1^{7}X_2 + \begin{bmatrix}46659\\59796\\38267\\39647\\27683\end{bmatrix} X_1^{6}X_2^{2} + \begin{bmatrix}32367\\23164\\64111\\63692\\29095\end{bmatrix} X_1^{5}X_2^{3} + \begin{bmatrix}37627\\25182\\59951\\60422\\11080\end{bmatrix} X_1^{4}X_2^{4} +$$

$$\begin{bmatrix}27200\\38476\\28698\\5708\\47718\end{bmatrix} X_1^{3}X_2^{5} + \begin{bmatrix}64271\\43542\\57950\\52276\\9739\end{bmatrix} X_1^{2}X_2^{6} + \begin{bmatrix}49159\\11328\\33520\\65039\\27178\end{bmatrix} X_1X_2^{7} + \begin{bmatrix}59456\\49518\\46071\\49716\\33760\end{bmatrix} X_2^{8} + \text{2069 terms}$$

5 equations
5 unknowns
Degree 16

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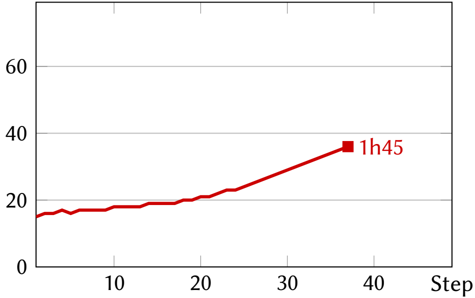
"Default" strategy:

- Irregular behavior
- Long calculation
- No complexity estimates

Degree of the polynomials at each step



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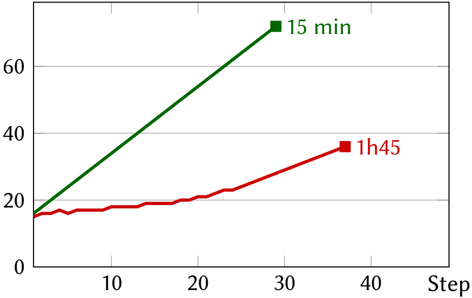
With weights:

= Subst. $X_i \leftarrow X_i^2$ ($i = 1 \ldots 4$):

- Regular behavior
- Faster calculation

Degree of the polynomials at each step



Weighted homogeneous: results and future works

Results:

- Full algorithmic strategy taking advantage of generic regularity properties
- Full understanding of the graduation (syzygy module, Hilbert series)
- Characterization of generic properties (regularity, semi-regularity, Noether position)
- Complexity bounds divided by $\left(\prod w_i\right)^3$
- Can be used by any existing implementation without computational cost

Future work:

- Automatic detection of the best system of weights
- More general structures allowing the weights to be 0 (elimination)...
- ... or < 0 (variables with local ordering, saturation)
- Multi-graduation: weighted homogeneous for several systems of weights (physics)

Tool for number theory: modern algorithms for Gröbner bases over rings
(With M. Francis)

- Applications:
  - Number theory [Lichtblau, 2011]
  - Lattice-based cryptography [Francis, Dukkipati 2016]
  - Computation in finitely-presented groups [Sims, 1994]
- Example: intersection of two ideals in $\mathbb{Z}[\sqrt{-11}][x, y] \simeq \mathbb{Z}[x, y, z]/\langle z^2 + 11\rangle$?

  Non Euclidean (non factorial) → $\mathbb{Z}[\sqrt{-11}]$

  Euclidean ($\mathbb{Z}$) → $\mathbb{Z}$

- Algorithms developed in the late 80's and early 90's
- Impossible to mitigate coefficient growth with modular methods
- Many usual criteria when coefficients are in a field become more complicated over rings
- Recent surge of interest with focus on $\mathbb{Z}$ and Euclidean rings (Lichtblau, Eder, Popescu...)

Gröbner bases over $\mathbb{Z}$: results and future work

Question:

- Signatures: technique for recovering and exploiting info. on the module of syzygies [Faugère, 2002]
- Is it possible to compute Gröbner bases with signatures over $\mathbb{Z}$?
- State of the art: No, impossible [Eder, Popescu 2017]

Results:

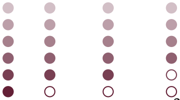
- New answer: Yes, with another definition!
- Proof of concept of two algorithms working over any principal ring
- Prototype implementation of the algorithms in Magma

Future work:

- Complete analysis of existing algorithms and criteria to identify what is or not possible
- Complexity analyses
- Competitive implementation of the algorithms

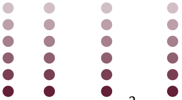
Tool for algebraic geometry: Gröbner bases over Tate algebras
(With X. Caruso and T. Vaccon)

- Tate series = convergent series over a complete valued ring (*e.g.* $\mathbb{Z}_p$ or $\mathbb{Q}_p$)
  $\iff$ the valuation of the coefficients goes to infinity
- Introduced by Tate in 1971 for rigid geometry
  (p-adic equivalent of the bridge between algebraic and analytic geometry over $\mathbb{C}$)
- No existing implementation of arithmetic or ideal operations



$$\sum_{i,j=0}^{\infty} p^{i+j} X^i Y^j = 1 + pX + pY + p^2 X^2 + \cdots$$

Tate series



$$\sum_{i=0}^{\infty} X^i = 1 + 1X + 1X^2 + 1X^3 + \cdots$$

Not a Tate series

Tate algebras: results and future work

Features of those systems:

- In the valued case, there is no difference between ring and field
- Main difficulty: in Tate series, we need to order terms (with coefficients)...
- ... in a mixed ordering: $pX < 1 < X$

Results:

- Definitions and algorithms for Gröbner bases over Tate algebras
- Implementation of arithmetic and Gröbner basis algorithms in Sage (included in Sage since version 8.5 [2019])
- Signature-based algorithms over Tate algebras

Future work:

- More efficient algorithms for reductions
- More optimized implementation

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Signature GB

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More criteria
Better implementation

Crypto., physics...

Weighted homogeneous

More general weights
Automatic detection

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Faster reductions

Structured — Polynomial systems — Extended

MRI, control

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Example of general questions: complexity and strategy for elimination?

Complexity and strategy for a system with finitely many solutions:

System → (Direct algo.) → Degree Gröbner basis → (Change of order) → Lexicographical Gröbner basis

Degree Gröbner basis — Maximal degree: Macaulay bound $\simeq \sum$ degrees

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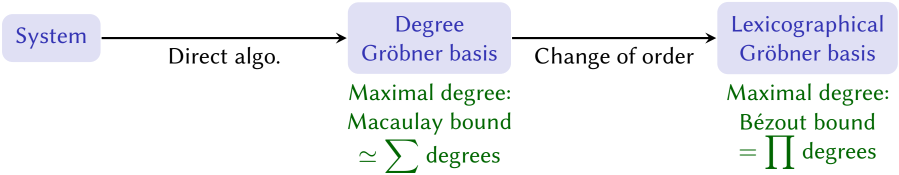
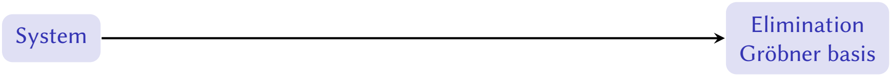


What about polynomial elimination?



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System → Elimination Gröbner basis

Elimination Gröbner basis — Maximal degree: $\ll$ Bézout bound

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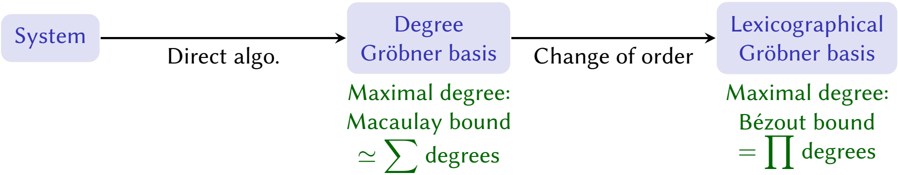

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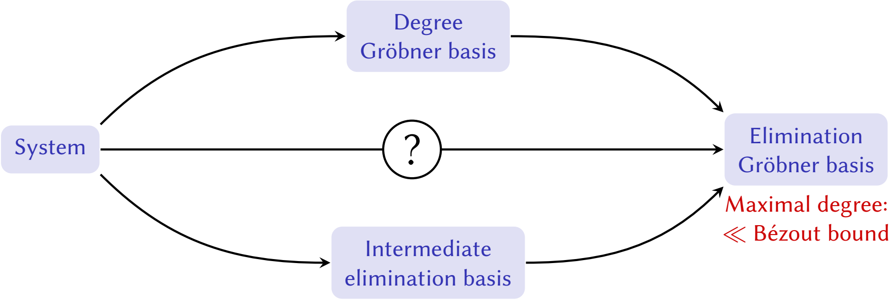

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