Gröbner bases over Tate algebras

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Precision and Gröbner bases

- Question: in $\mathbb{R}[X]$, reduce $f = X^2$ modulo $g = 0.01X - 1$

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- It terminates, but...

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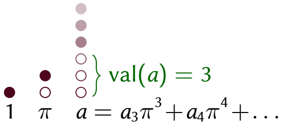
- This work: make sense of this process for convergent power series in $\mathbb{Z}_p[[X]]$

A recap on Complete Discrete Valuation Rings

- DVR = principal local domain K° with maximal ideal $\langle\pi\rangle$, residue field $\mathbb{F} = K^\circ/\langle\pi\rangle$

$\mathbb{Z}_p$	p	$\mathbb{F}_p$
$\mathbb{C}[[X]]$	X	$\mathbb{C}$

- Elements can be written $a = \sum_{n=0}^{\infty} a_n \pi^n$, $a_n \in \mathbb{F}$
- Valuation of a = max n such that π^n divides a
- Metric defined by "a is small $\iff$ val(a) is large"
- $\mathbb{Z}_p$ and $\mathbb{C}[[X]]$ are complete for this topology



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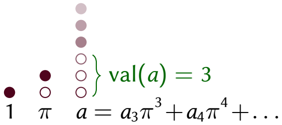

- No loss of precision possible:
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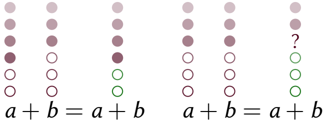

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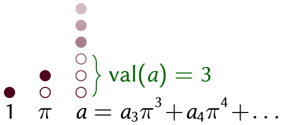

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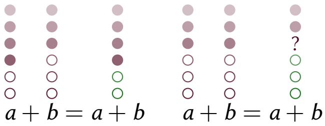
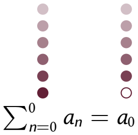


- In a CDVR, a series is convergent iff its general term tends to 0



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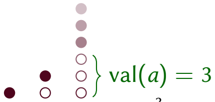

$1 \quad \pi \quad a = a_3\pi^3 + a_4\pi^4 + \dots$

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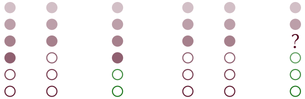

$a + b = a + b \qquad a + b = a + b$

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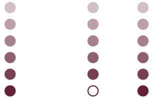

$\sum_{n=0}^{1} a_n = a_0 + a_1$

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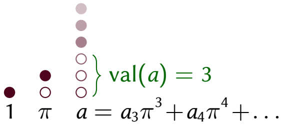

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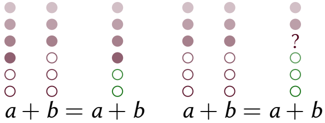

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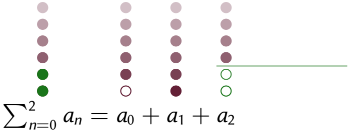

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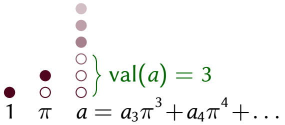

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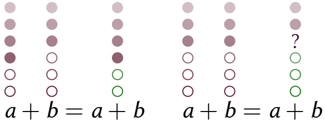

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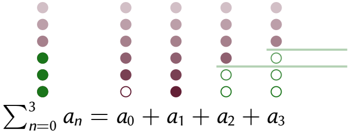

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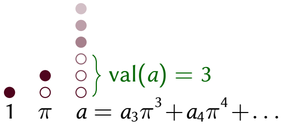

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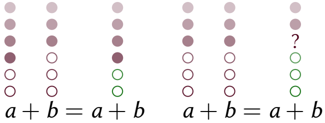

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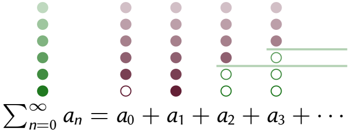


Tate Series

$\mathbf{X} = X_1, \ldots, X_n$

Definition

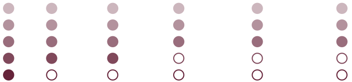
- $K\{\mathbf{X}\}^\circ$ = ring of series in $\mathbf{X}$ with coefficients in K° converging for all $\mathbf{x} \in K^\circ$
  = ring of power series whose general coefficients tend to 0

Motivation

- Introduced by Tate in 1971 for rigid geometry
  (p-adic equivalent of the bridge between algebraic and analytic geometry over $\mathbb{C}$)

Examples

- Polynomials (finite sums are convergent)



- $$\sum_{i,j=0}^{\infty} \pi^{i+j} X^i Y^j = 1 + \pi X + \pi Y + \pi^2 X^2 + \pi^2 XY + \pi^2 Y^2 + \cdots$$

- Not a Tate series: $\sum_{i=0}^{\infty} X^i = 1 + 1X + 1X^2 + 1X^3 + \cdots$

Term ordering for Tate algebras

$\mathbf{X}^{\mathbf{i}} = X_1^{i_1} \cdots X_n^{i_n}$

- Starting from a usual monomial ordering $1 <_m \mathbf{X}^{\mathbf{i}_1} <_m \mathbf{X}^{\mathbf{i}_2} <_m \dots$
- We define a term ordering putting more weight on large coefficients

Usual term ordering:

$$\pi \cdot 1 <_m 1\,\mathbf{X}^{\mathbf{i}_1} <_m \pi\,\mathbf{X}^{\mathbf{i}_2} <_m \pi^2\,\mathbf{X}^{\mathbf{i}_3} <_m \cdots$$

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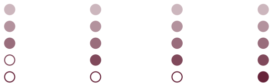

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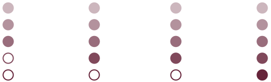

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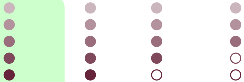
- It has infinite descending chains, but they converge to zero
- Tate series always have a leading term

$\mathrm{LT}(f)$



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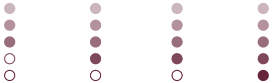

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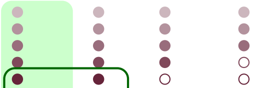

$$f = \boxed{a_2 XY + a_1 X} + a_0 \cdot 1 + a_3 X^2 Y^2 + \ldots$$

$$\bar{f} = \overline{a_2} XY + \overline{a_1} X$$

- Isomorphism $\quad K\{\mathbf{X}\}^\circ / \langle \pi \rangle \quad \simeq \quad \mathbb{F}[\mathbf{X}]$

$$f \quad \mapsto \quad \bar{f}$$

compatible with the term order

Gröbner bases

- Standard definition once the term order is defined:

$$G \text{ is a Gröbner basis of } I \iff \text{for all } f \in I, \text{ there is } g \in G \text{ s.t. } \mathrm{LT}(g) \text{ divides } \mathrm{LT}(f)$$

- Standard equivalent characterizations:
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- Every Tate ideal has a finite Gröbner basis
- It can be computed using the usual algorithms (reduction, Buchberger, F_4)
- In practice, the algorithms run with finite precision and without loss of precision

No division by π

How does it work? (4 $\implies$ 3)

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   $\overline{G}$ is a Gröbner basis of $\overline{I}$

$$\overline{f} - q_1\overline{g_1} - q_2\overline{g_2} - \cdots - q_r\overline{g_r} = 0$$

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4. So we have a sequence of reductions

$$f - \sum_{i=1}^{r} q_i g_i = f - \sum_{i=1}^{r} q_i \overline{g_i} + \sum_{i=1}^{r} q_i \left(\overline{g_i} - g_i\right)$$

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$$\overline{f} - q_1\overline{g_1} - q_2\overline{g_2} - \cdots - q_r\overline{g_r} = 0$$

4. So we have a sequence of reductions

$$f - \sum_{i=1}^{r} q_i g_i = f - \sum_{i=1}^{r} q_i \overline{g_i} + \sum_{i=1}^{r} q_i \left(\overline{g_i} - g_i\right)$$

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How does it work? ($4 \implies 3$)

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What about Tate series over a field?

- CDVF = fraction field K of a CDVR K°

  $\mathbb{Q}_p$ $\mathbb{Z}_p$

  $\mathbb{C}((X))$ $\mathbb{C}[[X]]$

- Elements can be written $a = \sum_{n=-r}^{\infty} a_n \pi^n$, $a_n \in \mathbb{F}$
- The valuation can be negative but not infinite
- Same metric, same topology as K°

$$a = a_{-3}\pi^{-3} + a_{-2}\pi^{-2} + \dots$$

$\}$ $\mathrm{val}(a) = -3$

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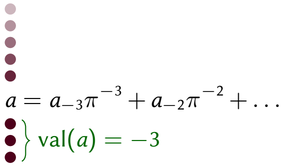

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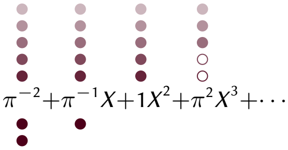

- Tate series can be defined as in the integer case
- Same order, same definition of Gröbner bases
- Main difference: πX now divides X

- Another surprising equivalence
  1. G is a normalized GB of I
  2. $G \subset K\{\mathbf{X}\}^\circ$ is a GB of $I \cap K\{\mathbf{X}\}^\circ$

$\forall g \in G$, $\mathrm{LC}(g) = 1$ (in part., $G \subset K\{\mathbf{X}\}^\circ$)

- In practice, we emulate computations in $K\{\mathbf{X}\}^\circ$ in order to avoid losses of precision

Generalizing the convergence condition: log-radii in $\mathbb{Z}^n$

$\mathbf{X}^{\mathbf{i}} = X_1^{i_1} \cdots X_n^{i_n}$

Definition

- $K\{\mathbf{X}\}$ = ring of power series converging for all $\mathbf{x} \in K^\circ$
  = ring of power series whose general coefficients tend to 0
  = ring of power series $\sum a_{\mathbf{i}} \mathbf{X}^{\mathbf{i}}$ with $\mathrm{val}(a_{\mathbf{i}}) \xrightarrow[|\mathbf{i}| \to \infty]{} +\infty$

$f \notin K\{X\}$

$$f(X) = \sum_{i=0}^{\infty} X^i = 1 + 1X + 1X^2 + \cdots \longrightarrow f(x) = 1 + x + x^2 + \cdots \text{ is divergent}$$

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- Reduction to previous case by change of variables: $f(\pi X) = 1 + \pi X + \pi^2 X^2 + \cdots$

Generalizing the convergence condition: log-radii in $\mathbb{Q}^n$

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Definition

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- The term order is not the same!

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$f \in K\{X; 1\}$

Log-radii in $\mathbb{Q}^n$ are more complicated, but things still work.

convergent

- Reduction to previous case by change of variables: $f(\pi X) = 1 + \pi X + \pi^2 X^2 + \cdots$

Conclusion and perspectives

What we presented here

- Tate series = formal power series appearing in algebraic geometry
- Definitions of Gröbner bases for Tate series
- Algorithms for computing and using those Gröbner bases
- Data structure and algorithms implemented in Sage (version 8.5, 22/12/2018)

Extensions

- Coefficients in a complete discrete valuation field (controlling the precision)
- Tate series with convergence radius different from 1 (integer or rational log)

Perspectives

- Faster reduction: algorithms for local monomial orderings and standard bases (Mora)
- Faster Gröbner basis computation: signature-based algorithms

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Thank you for your attention!

More information and references:

- Xavier Caruso, Tristan Vaccon and Thibaut Verron (2019). 'Gröbner bases over Tate algebras'. In: *ISSAC'19*, arXiv:1901.09574. arXiv: 1901.09574 [math.AG]