Algebraic classification methods for an optimal control problem in medical imagery

Bernard Bonnard[1,2] Olivier Cots[5] Jean-Charles Faugère[3]
Alain Jacquemard[1] Jérémy Rouot[4] Mohab Safey El Din[3]
Thibaut Verron[6]

1. Université de Bourgogne-Franche Comté, Dijon
2. Inria Sophia Antipolis, Équipe McTAO
3. UPMC Paris Sorbonne Universités, Inria Paris, CNRS, LIP6, Équipe PolSys
4. EPF-Écoles d'Ingénieurs, Troyes
5. Toulouse Universités, INP-ENSEEIHT-IRIT, CNRS, Équipe APO
6. Institute for Algebra, Johannes Kepler University, Linz

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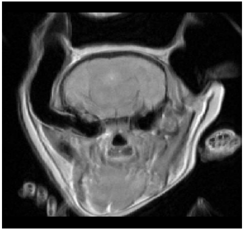
Contrast optimization for MRI

(N)MRI = (Nuclear) Magnetic Resonance Imagery

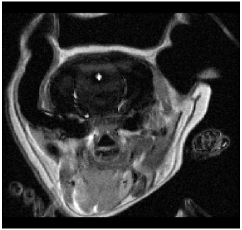
1. Apply a magnetic field to a body
2. Measure the radio waves emitted in reaction

Goal = optimize the contrast = distinguish two biological matters from this measure

Example: *in vivo* experiment on a mouse brain (brain vs parietal muscle)[1]



Bad contrast (not enhanced)



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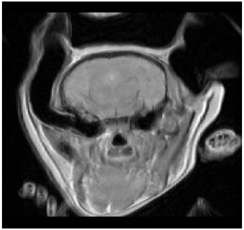

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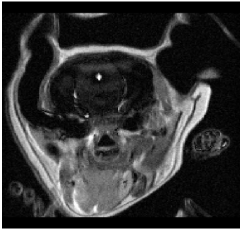

Good contrast (enhanced)

Known methods:

- inject contrast agents to the patient: potentially toxic...
- enhance the contrast dynamically $\Longrightarrow$ optimal control problem

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Problem and results

Study of optimal control strategy for the MRI

- Optimal control theory: find settings for the MRI device ensuring e.g. good contrast
- Already proved to give better results than implemented heuristics[2]
- Powerful tools allow to understand the control policies

These questions reduce to algebraic problems

- Invariants of a group action on vector fields
- Algebraic: rank conditions, polynomial equations, eigenvalues...

Contribution: algebraic tools for this workflow

- Demonstrate use of existing tools
- Dedicated strategies for specific problems (real roots classification) adapted to the structure of the systems (determinantal systems)
- These structures extend beyond the MRI problem

[2] Marc Lapert, Yun Zhang, Martin A. Janich, Steffen J. Glaser and Dominique Sugny (2012). 'Exploring the Physical Limits of Saturation Contrast in Magnetic Resonance Imaging'. In: *Scientific Reports* 2.589.

Outline of the talk

1. Context and problem statement
    - Magnetic Resonance Imagery
    - Physical modelization of the problem
2. Optimal control theory
    - What is control theory?
    - Pontryagin's Maximum principle
    - Study of singular extremals: algebraic questions
3. General algebraic techniques
    - Tools for polynomial systems
    - Examples of results
4. Real roots classification for the singularities of determinantal systems
    - What is the goal?
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5. Conclusion

The Bloch equations for a single spin

The Bloch equations

$$\begin{cases} \dot{y} &= -\Gamma y - uz \\ \dot{z} &= \gamma(1-z) + uy \end{cases} \quad \rightsquigarrow \quad \dot{q} = F(\gamma, \Gamma, q) + uG(q)$$

- $q = (y, z)$: state variables
- γ, Γ: relaxation parameters (constants depending on the biological matter)
- u: control function (the unknown of the problem)

Physical limitations

- State variables: the Bloch Ball

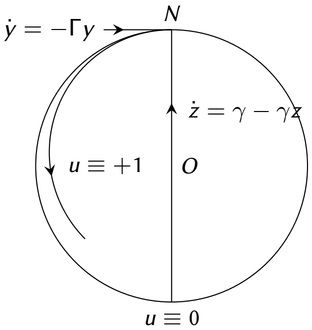
$$y^2 + z^2 \leq 1$$

- Parameters:

$$2\gamma \geq \Gamma > 0$$

- Control:

$$-1 \leq u \leq 1$$



Optimal control problems

Bloch equations for 2 spins: $\begin{cases} \dot{q}_1 = & F_1(\gamma_1, \Gamma_1, q_1) + uG_1(q_1) \\ \dot{q}_2 = & F_2(\gamma_2, \Gamma_2, q_2) + uG_2(q_2) \end{cases}$

Contrast problem

- Two matters, 4 parameters $\gamma_1, \Gamma_1, \gamma_2, \Gamma_2$
- Both spins have the same dynamic: $F_1 = F_2 = F$, $G_1 = G_2 = G$
- Equations

$$\begin{cases} \dot{q}_1 & = F(\gamma_1, \Gamma_1, q_1) + uG(q_1) \\ \dot{q}_2 & = F(\gamma_2, \Gamma_2, q_2) + uG(q_2) \end{cases}$$

- Goal: saturate #1, maximize #2:

$$\begin{cases} y_1 = z_1 = 0 \\ \text{Maximize } |(y_2, z_2)| \end{cases}$$

Multi-saturation problem

- Two spins of the same matter: $\Gamma_1 = \Gamma_2 = \Gamma$, $\gamma_1 = \gamma_2 = \gamma$
- Small perturbation on the second spin: $F_1 = F_2 = F$, $G_2 = (1 - \varepsilon)G_1$
- 2 parameters + ε
- Equations:

$$\begin{cases} \dot{q}_1 & = F(\gamma, \Gamma, q_1) + uG(q_1) \\ \dot{q}_2 & = F(\gamma, \Gamma, q_2) + u(1 - \varepsilon)G(q_2) \end{cases}$$

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What is control theory?

Controlled system:

$$\begin{cases} \dot{q}(t) = \mathcal{F}(q(t), u(t), t) \\ q(0) = q_0 \quad \text{(or more general constraints)} \end{cases}$$

Control theory = answer one of the following questions for such a system:

Accessibility Given V, is there a control u such that $q(t_f) \in V$ for some t_f? More generally, what is the set of accessible points?

Optimality What is the shortest time in which we can reach a target set? More generally, minimize a combination of a running cost (Lagrange cost, *e.g.* energy):

$$C_L(u) = \int_0^{t_f} f^0(q(t), u(t), t)\mathrm{d}t$$

and a terminal cost (Mayer cost, *e.g.* time):

$$C_M(u) = g(q(t_f), t_f)$$

Interlude 1: different settings, different problems?

$$\begin{cases} \dot{q}(t) = \mathcal{F}(q(t), u(t), t) \\ q(0) = q_0 \end{cases}$$

Many different settings...

- Is the system autonomous? (Or does $\mathcal{F}$ depend on t?)
- Is the cost evaluated at t_f, or all along the trajectory?

... and two kinds of questions

- What is the reachable set?
- What is the cost of reaching a given set?

All of these questions are actually equivalent!

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Pontryagin's Maximum principle

Optimal control problem: minimize $\int f^0(q, u)$ under the constraint $\dot{q} = F(q, u)$ ($q(t) \in \mathbb{R}^n$)

For finite-dimensional optimization, we have Lagrange multipliers.
For control theory, we have...

Hamiltonian

Introduce multipliers $p = (p_1, \ldots, p_n) : \mathbb{R} \to \mathbb{R}^n$.
The Hamiltonian associated with the control problem is defined as

$$H(q, p, u) := \langle p, F(q, u) \rangle - f^0(q, u)$$

Pontryagin's Maximum principle

If u is an optimal control, then q, p and u are solutions of

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases}$$

and almost everywhere in t, $u(t)$ maximizes the Hamiltonian:

$$H(q(t), p(t), u(t)) = \max_{v \in [-1,1]} H(q(t), p(t), v)$$

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The affine case: bang and singular arcs

The Bloch equations form an affine control problem:

$$\dot{q} = F(q) + uG(q)$$

Pontryagin's principle, the affine case

The control u maximizes over $[-1, 1]$:

$$H(q, p, u) = H_F(q, p) + uH_G(q, p).$$

Two situations:

- $H_G \neq 0 \implies u = \mathrm{sign}(H_G)$: "Bang" arc
- $H_G = 0 \implies$???

Singular trajectories for the Bloch equations

They satisfy $\dot{q} = D\, F(q) - D'\, G(q)$ with optimal control $u = \dfrac{D'}{D}$.

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  In practice one chooses u such that H_G remains 0: Singular arc

  $\implies$ need bifurcation strategies...

Singular trajectories for the Bloch equations

They satisfy $\dot{q} = D\,F(q) - D'\,G(q)$ with optimal control $u = \dfrac{D'}{D}$.

What are D and D'?

For singular trajectories, we have 4 linear equations in p:

$$p \cdot F = 0 \qquad \text{“Exceptional” case}$$
$$p \cdot G = 0 \qquad \text{Singular arc}$$
$$p \cdot [G, F] = 0 \qquad \mathrm{d}/\mathrm{d}t \text{ the previous one...}$$
$$p \cdot ([[G, F], F] + u[[G, F], G]) = 0 \qquad \text{... and again}$$

Cramer's rule: the solution is

$$u = \frac{D'}{D} \quad \text{(exists iff } D \neq 0\text{)}$$

where

$$D = \det(G, F, [G, F], [[G, F], G])$$
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D and D' are determinants of 4×4 matrices

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Interlude 2: Lie brackets and controllability of affine systems

Affine control problem: $\dot{q} = uF + vG$ (simpler case, no drift)

Question: What is $[F, G] = \dfrac{\partial G}{\partial q} F - \dfrac{\partial F}{\partial q} G$?

It measures the interactions between the two controls!

Theorem (characterization of controllability)

The system is controllable at q (= we can go in any direction from q) if and only if

$$\dim \operatorname{Vect}(F, G, [F, G], [[F, G], F], \dots) = \text{dimension of the ambient space}$$

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Study of invariants

Group action on vector fields (F, G)

$$\text{Control system: } \dot{q} = F(q) + uG(q)$$

- Changes of coordinates: $q \leftarrow \varphi(q)$
- Feedback: $u \leftarrow \alpha(q) + \beta(q)v$

Long-term goal: classification of the parameters via invariants of this group action

Example: control of a single spin[3]

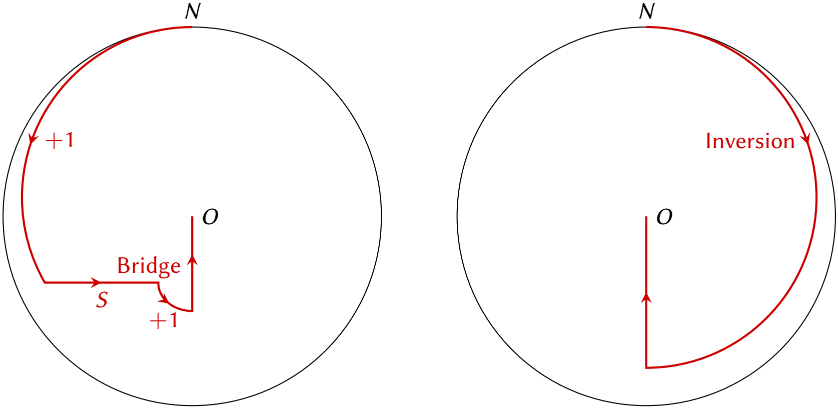

Figure: Time-minimal saturation for a single spin: left: $2\Gamma < 3\gamma$, right: $2\Gamma \geq 3\gamma$

[3] Marc Lapert (2011). 'Développement de nouvelles techniques de contrôle optimal en dynamique quantique : de la Résonance Magnétique Nucléaire à la physique moléculaire'. PhD thesis. Université de Bourgogne, Dijon, France.

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Examples of invariants (fixed values of the parameters)

- Hypersurface $\Sigma : \{D = 0\}$
- Singularities of Σ
- Set where F and G are colinear
- Set where G and $[F, G]$ are colinear
- Equilibrium points: $\{D = D' = 0\}$
- Eigenvalues of the linearized system at equilibrium points (up to a constant)
- ...

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Workflow

Control problem

↓ Pontryagin

Description of extremal trajectories

↓

Classification of algebraic invariants of the singular dynamics

↓

Classification of the trajectories

$$\dot{q} = F(\gamma, \Gamma, q) + uG(q)$$

- Bang arcs: $u \equiv \pm 1$
- Singular arcs: $u = \dfrac{D'}{D}$

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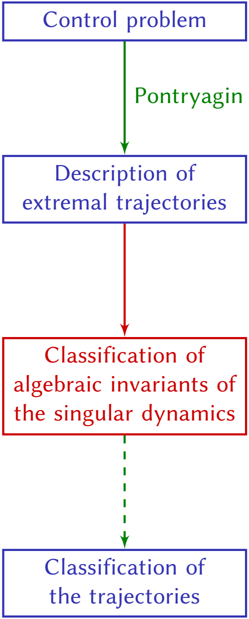

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Problem: study and classification of the solutions of systems of polynomial equations

Method: exact algorithmic tools

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Polynomial tools: factorization and elimination

Factorization

- Given $P \in \mathbb{Q}[X_1, \ldots, X_n]$, compute $F_i \in \mathbb{Q}[X_1, \ldots, X_n], \alpha_i \in \mathbb{N}$ such that $P = F_1^{\alpha_1} \cdots F_r^{\alpha_r}$
- Very fast, efficiently implemented in most CAS
- Ex. square-free form: $\sqrt{P} := F_1 \cdots F_r$ has the same zeroes as P

Elimination

- Given an ideal $I \subset \mathbb{Q}[X_1, \ldots, X_n]$ and $k \in \{1, \ldots, n\}$, compute $I \cap \mathbb{Q}[X_{k+1}, \ldots, X_n]$
- Computationally expensive, many different tools: resultants, Gröbner bases...
- Ex. saturation: $\langle f_1, \ldots, f_r\rangle : f^\infty = \langle f_1, \ldots, f_r, Uf - 1\rangle \cap \mathbb{Q}[X_1, \ldots, X_n]$
  The roots of this system "are" the roots of $f_1, \ldots, f_r$, minus the zeroes of f

Typical example of simplification

If I contains $P = fg$, we can split the study into:

1. the roots of $I + \langle f\rangle$
2. the roots of $I + \langle g\rangle$ saturated by f

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  The roots of this system "are" the roots of $f_1, \ldots, f_r$, minus the zeroes of f

Typical example of simplification

If I contains $P = fg$, we can split the study into:

1. the roots of $I + \langle f \rangle$
2. the roots of $I + \langle g \rangle$ saturated by f

Examples for multi-saturation

$$\begin{cases} \dot{q}_1 & = D\,F(\gamma, \Gamma, q_1) - D'\,G(q_1) \\ \dot{q}_2 & = D\,F(\gamma, \Gamma, q_2) - D'\,(1-\varepsilon)G(q_2) \end{cases}$$

Singularities of $\{D = 0\}$

- North pole
- Line defined by $\begin{cases} y_1 = (1-\varepsilon)y_2 \\ z_1 = z_2 = z_S := \frac{\gamma}{2(\Gamma-\gamma)} \end{cases}$ (cf. the horizontal line for a single spin)

Equilibrium points $D = D' = 0$

- Horizontal plane $z_1 = z_2 = z_S = \frac{\gamma}{2(\Gamma-\gamma)}$
- Vertical line $y_1 = y_2 = 0$, $z_1 = z_2$
- 3 more complicated surfaces (related to the colinearity loci)

We can fully describe all invariants!

Previous results for the contrast problem[4]

Study of 4 experimental cases:

Matter #1 / # 2	γ_1	Γ_1	γ_2	Γ_2
Water / cerebrospinal fluid	0.01	0.01	0.02	0.10
Water / fat	0.01	0.01	0.15	0.31
Deoxygenated / oxygenated blood	0.02	0.62	0.02	0.15
Gray / white brain matter	0.03	0.31	0.04	0.34

Separated by means of several invariants:

- Number of singularities of $\{D = 0\}$
- Structure of $\{D = D' = 0\}$
- Eigenvalues of the linearizations at equilibrium points
- Study of the quadratic approximations at points where the linearization is 0

[4] Bernard Bonnard, Monique Chyba, Alain Jacquemard and John Marriott (2013). 'Algebraic geometric classification of the singular flow in the contrast imaging problem in nuclear magnetic resonance'. In: *Mathematical Control and Related Fields* 3.4, pp. 397–432. ISSN: 2156-8472. DOI: 10.3934/mcrf.2013.3.397.

Classification for the contrast problem

$$\begin{cases} \dot{q}_1 &= D\, F(\gamma_1, \Gamma_1, q_1) - D'\, G(q_1) \\ \dot{q}_2 &= D\, F(\gamma_2, \Gamma_2, q_2) - D'\, G(q_2) \end{cases}$$

More complicated

- 4 variables, 4 parameters ($\rightsquigarrow$ 3 by homogeneity)
- Polynomials of high degree

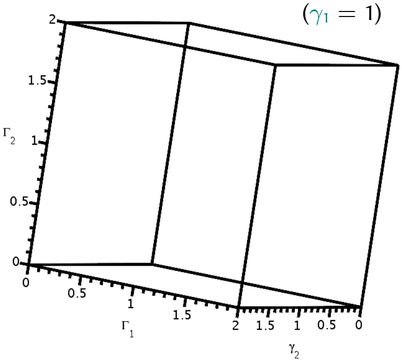
Singularities of $\{D = 0\}$ using Gröbner bases and factorisations/saturations

(After appropriate saturations) the ideal contains

$$\begin{cases} 0 &= P_{y_2}(y_2^2, \bullet) \text{ with degree 4 in } y_2^2 \text{ (8 roots)} \\ \bullet y_1 &= P_{y_1}(y_2, \bullet) \\ \bullet z_1 &= P_{z_1}(y_2, \bullet) \\ \bullet z_2 &= P_{z_2}(y_2, \bullet) \\ & \vdots \end{cases}$$

Study of the leading coefficient and discriminant of P_{y_2}

Singularities of $\{D = 0\}$ for the contrast problem: first results



Properties:

- Finite number of singularities
  for each value of the parameters
- Singularities come in pairs:
  invariant under $(y_i \mapsto -y_i)$

Classification in terms of Γ_i, γ_i:

- Generically: 4 pairs of singularities
- 3 pairs on a surface
  with several components:
  - one hyperplane
  - one quadric
  - one degree 24 surface
  - ...
- 2 pairs on a curve with many components
- 1 pair on a set of points

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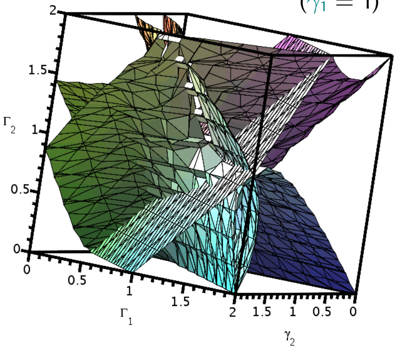

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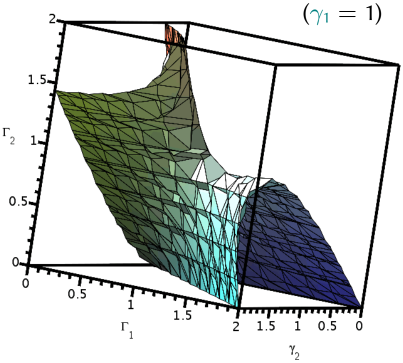

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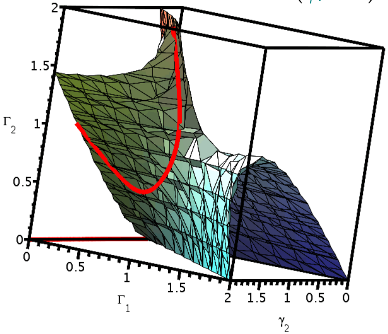
$(\gamma_1 = 1)$



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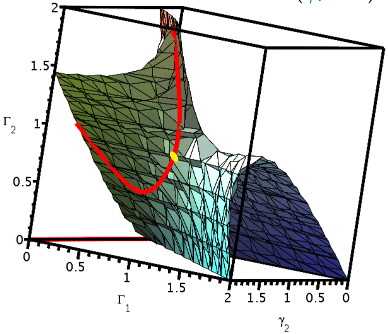

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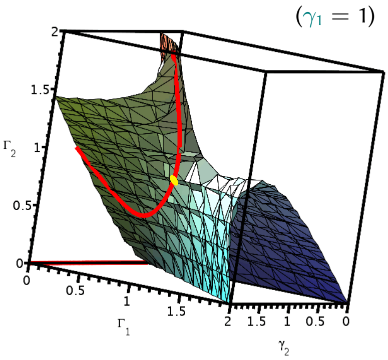

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Can we get more information? For example, information about real points?

Outline of the talk

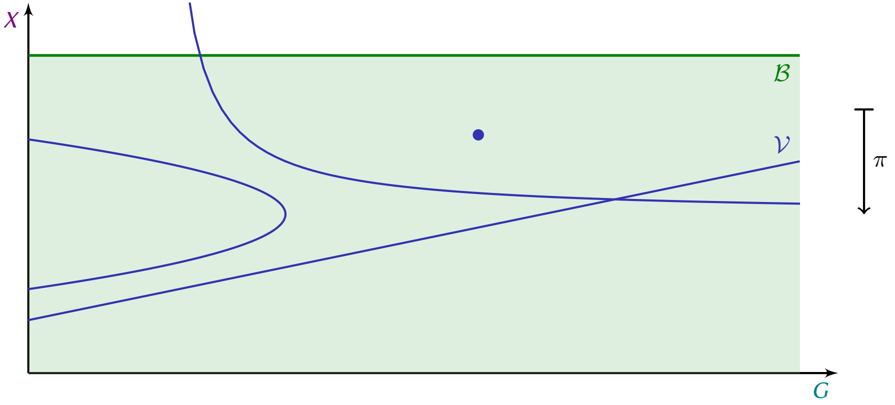
1. Context and problem statement
   - Magnetic Resonance Imagery
   - Physical modelization of the problem

2. Optimal control theory
   - What is control theory?
   - Pontryagin's Maximum principle
   - Study of singular extremals: algebraic questions

3. General algebraic techniques
   - Tools for polynomial systems
   - Examples of results

4. Real roots classification for the singularities of determinantal systems
   - What is the goal?
   - State of the art and main results
   - General strategy: what do we need to compute?
   - Dedicated strategy for determinantal systems
   - Results for the contrast problem

5. Conclusion



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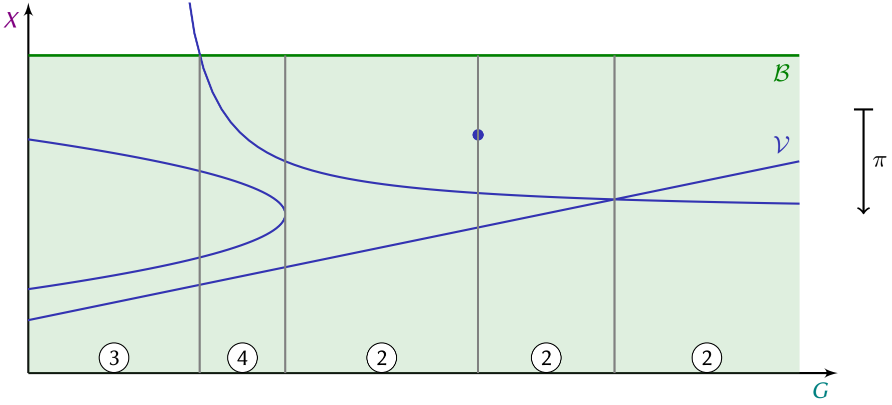
- Semi-algebraic constraints $\mathcal{B}$: Bloch Ball $y_i^2 + z_i^2 - 1 \leq 0$

The goal : real roots classification



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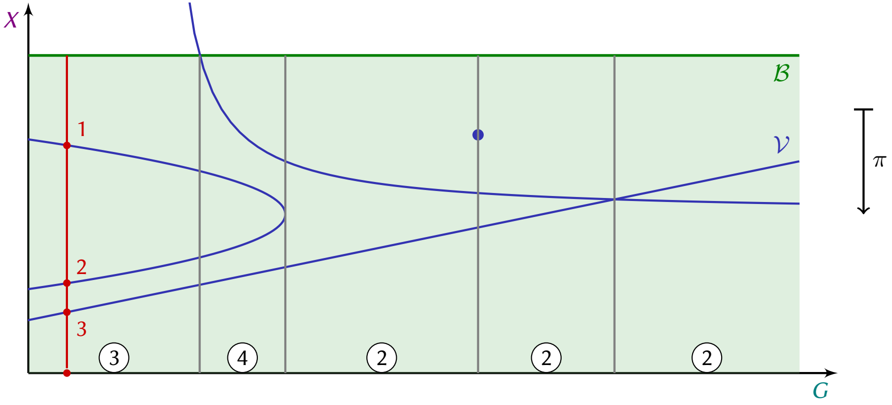
Partition of the parameter space depending on the number of points of $\mathcal{V} \cap \mathcal{B}$ above

The goal : real roots classification



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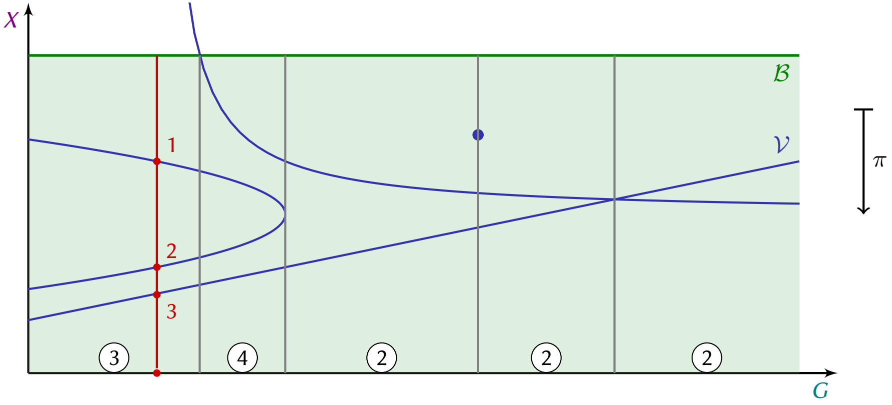

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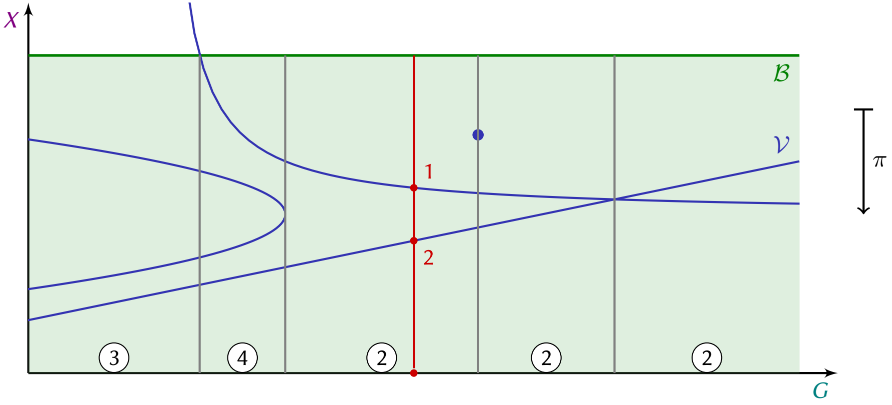

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State of the art and main results

State of the art:

- General tool: Cylindrical Algebraic Decomposition
  Collins, 1975
- Specific tools for roots classification
  Yang, Hou, Xia, 2001
  Lazard, Rouillier, 2007

Problem

- None of these algorithms can solve the problem efficiently:
  - 1050 s in the case of water ($\gamma_1 = \Gamma_1 = 1 \to 2$ parameters)
  - $>$ 24 h in the general case (3 parameters)
- Can we exploit the determinantal structure to go further?

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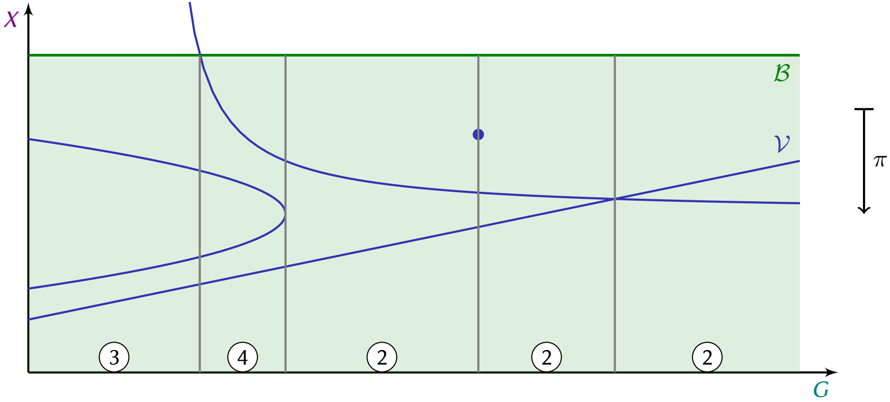
Main results

Bonnard, Faugère, Jacquemard, Safey, V., 2016

- Dedicated strategy for real roots classification for determinantal systems
- Can use existing tools for elimination
- Main refinements:
  - Rank stratification
  - Incidence varieties

- Faster than general algorithms:
  - 10 s in the case of water
  - 4 h in the general case
- Results for the application
  - Full classification
  - In the case of water: 1, 2 or 3 singularities
  - In the general case: 1, 2, 3, 4 or 5 singularities

General strategy for the real roots classification problem



In our case, the only points where the number of roots may change are projections of:

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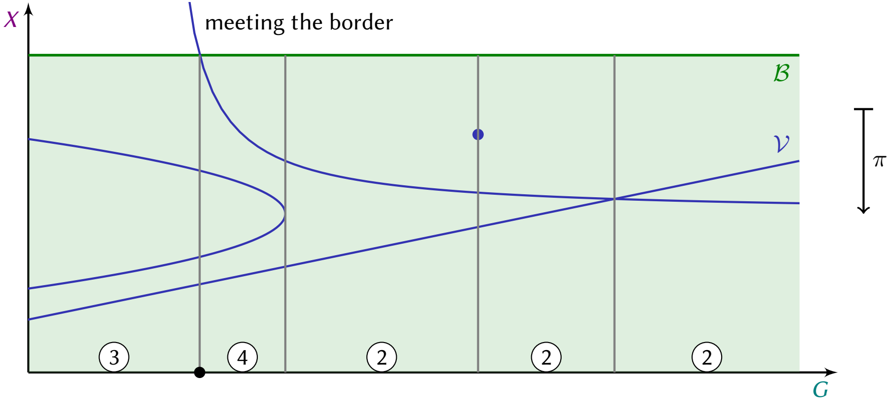

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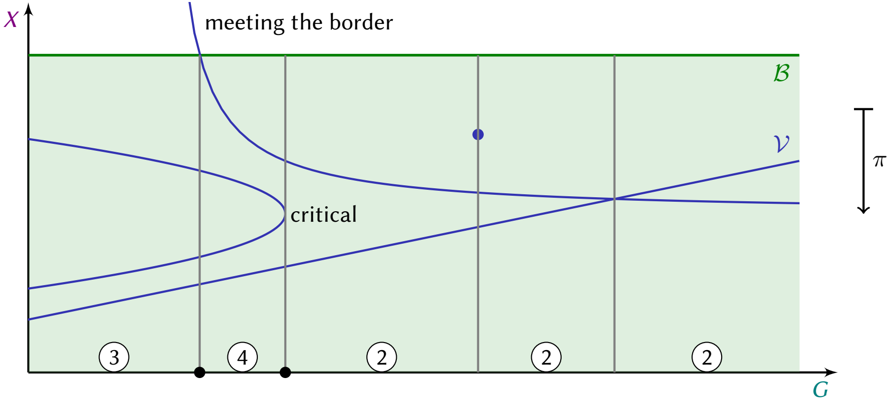

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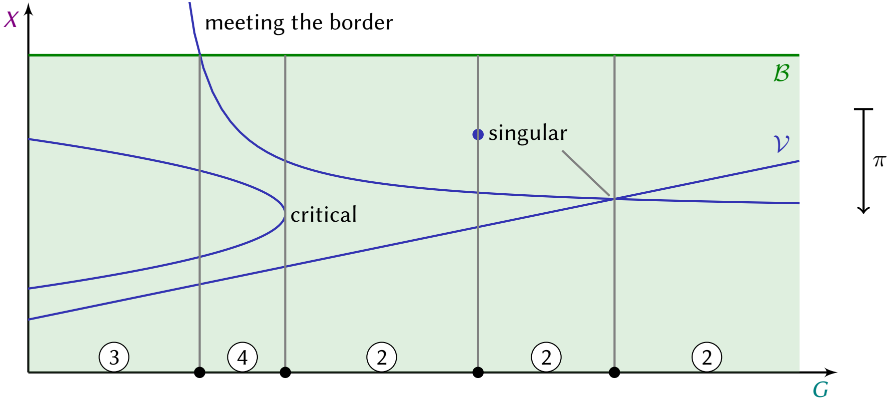

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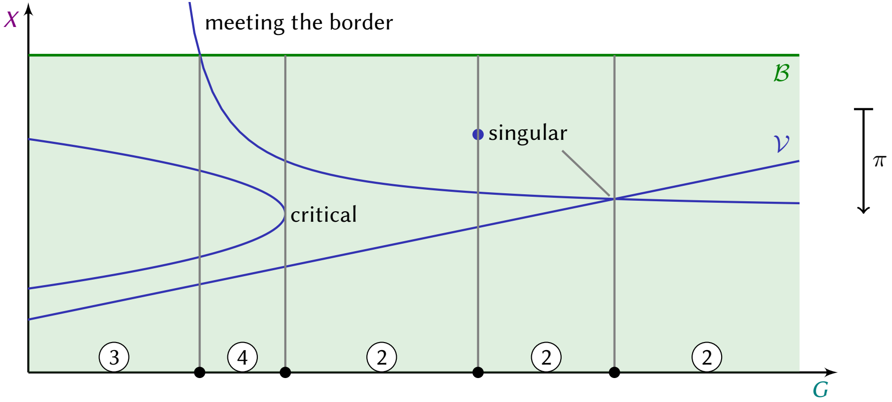

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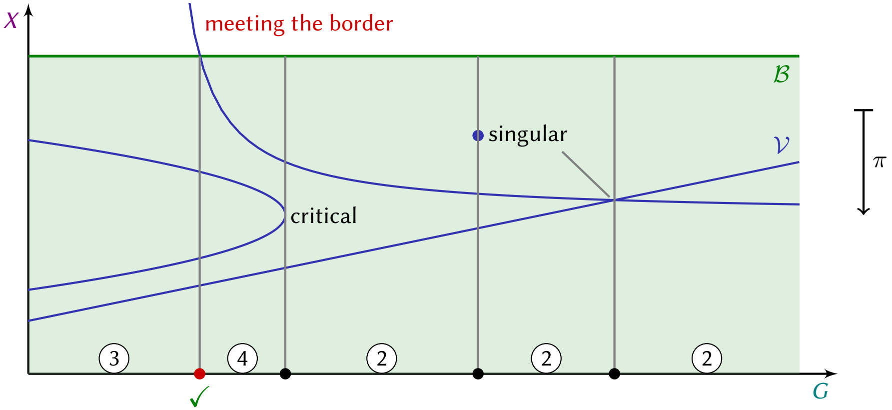
$\left.\begin{array}{l}\text{critical points of } \pi \text{ restricted to } \mathcal{V}\\ \text{singular points of } \mathcal{V}\end{array}\right\} =: K(\pi, \mathcal{V})$

We want to compute $P \in \mathbb{Q}[G]$ with $P \neq 0$ and P vanishing at all these points

General strategy for the real roots classification problem

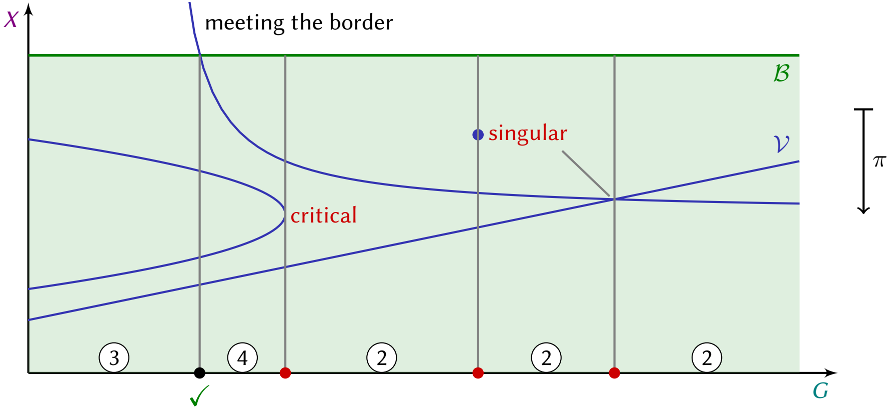


Intersection with the border

For each inequality $f > 0$ defining $\mathcal{B}$

1. Add $f = 0$ to the equations of $\mathcal{V}$
2. Compute the image of the variety through π (eliminate **X**)

General strategy for the real roots classification problem



Critical and singular points

$$(\mathbf{X}, \mathbf{G}) \in K(\pi, \mathcal{V})$$
$$\iff \text{Jac}(F, \mathbf{X}) \text{ has rank} < d$$

Requirements

- F generates the ideal of $\mathcal{V} \implies$ radical
- $\mathcal{V}$ is equidimensional with codimension d

Properties of determinantal systems

Determinantal systems

- $A = k \times k$-matrix filled with polynomials in n variables $\mathbf{X}$ and t parameters $\mathbf{G}$
- $1 \leq r < k$ target rank
- Determinantal variety: $V_{\leq r}(A) = \{(\mathbf{x}, \mathbf{g}) : \mathrm{rank}(A(\mathbf{x}, \mathbf{g})) \leq r\}$

Our system: $\mathcal{V} = \{D = \frac{\partial D}{\partial y_1} = \frac{\partial D}{\partial y_2} = \frac{\partial D}{\partial z_1} = \frac{\partial D}{\partial z_2} = 0\}$

$\Longrightarrow$ In terms of determinantal systems: $n = 4$, $k = 4$, $r = 3$, $\mathcal{V} = K(\pi, V_{\leq r}(M))$

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For a generic matrix A with the same parameters

- $V_{\leq r}(A)$ equidimensional with codimension $(k - r)^2$
- $\operatorname{Sing}(V_{\leq r}(A)) = V_{\leq r-1}(A)$, t-equidimensional
- $\operatorname{Crit}(\pi, V_{\leq r}(A))$ has dimension $< t$
- Natural stratification : $K(\pi, V_{\leq r}(A)) = \operatorname{Sing}(V_{\leq r}(A)) \cup \operatorname{Crit}(\pi, V_{\leq r}(A))$

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For our specific matrix M

- $V_{\leq r-1}(M) \subset \mathcal{V}$ (always true: why?)
- $V_{\leq r-1}(M)$ is equidimensional with dimension t
- $\mathcal{V} \setminus V_{\leq r-1}(M)$ has dimension $< t$
- Rank stratification : $\mathcal{V} = (\mathcal{V} \cap V_{\leq r-1}(M)) \cup (\mathcal{V} \setminus V_{\leq r-1}(M))$

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Rank stratification

$$
\begin{tikzcd}
\boxed{\mathcal{V} = K(\pi, V_{\leq r}(M))} \arrow[d, dashed] \\
\boxed{K(\pi, \mathcal{V})} \arrow[d, dashed] \\
\boxed{\begin{array}{c} P \in \mathbb{C}[\mathbf{G}] \\ \text{s.t. } \pi(K(\pi, \mathcal{V})) \subset V(P) \end{array}}
\end{tikzcd}
$$

Rank stratification

$$\mathcal{V} = K(\pi, V_{\leq r}(M))$$

(M has rank exactly r)

$$\mathcal{V}_1 := \mathcal{V} \cap V_{=r}(M)$$

(M has rank $< r$)

$$\mathcal{V}_2 := \mathcal{V} \cap V_{\leq r-1}(M)$$

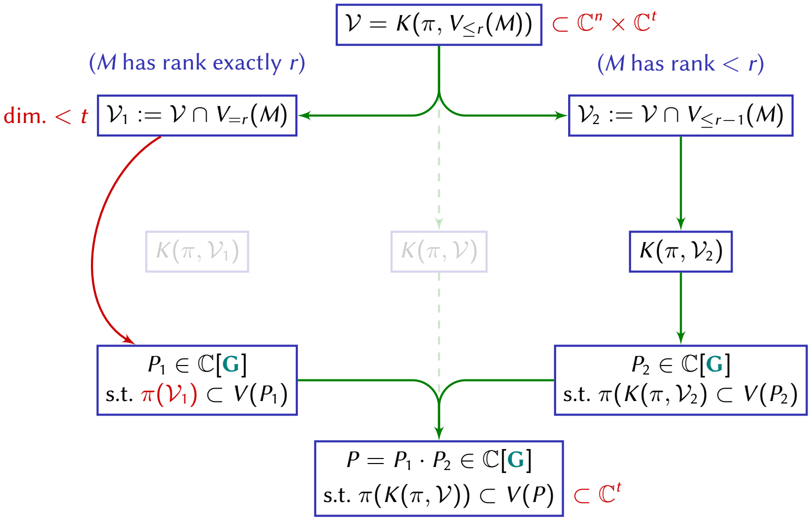
$$K(\pi, \mathcal{V}_1) \qquad K(\pi, \mathcal{V}) \qquad K(\pi, \mathcal{V}_2)$$

$P_1 \in \mathbb{C}[\mathbf{G}]$
s.t. $\pi(K(\pi, \mathcal{V}_1)) \subset V(P_1)$

$P_2 \in \mathbb{C}[\mathbf{G}]$
s.t. $\pi(K(\pi, \mathcal{V}_2) \subset V(P_2)$

$P = P_1 \cdot P_2 \in \mathbb{C}[\mathbf{G}]$
s.t. $\pi(K(\pi, \mathcal{V})) \subset V(P)$

Rank stratification



Rank stratification

$\boxed{\mathcal{V} = K(\pi, V_{\leq r}(M))} \subset \mathbb{C}^n \times \mathbb{C}^t$

(M has rank exactly r)

(M has rank $< r$)

dim. $< t$ $\boxed{\mathcal{V}_1 := \mathcal{V} \cap V_{=r}(M)}$

$\boxed{\mathcal{V}_2 := \mathcal{V} \cap V_{\leq r-1}(M)} = V_{\leq r-1}(M)$

$K(\pi, \mathcal{V}_1)$

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s.t. $\pi(K(\pi, V_{\leq r-1}(M)) \subset V(P_2)$

$P = P_1 \cdot P_2 \in \mathbb{C}[\mathbf{G}]$
s.t. $\pi(K(\pi, \mathcal{V})) \subset V(P)$ $\subset \mathbb{C}^t$

Modelization of $V_{=r}$

We can model $V_{\leq r}$ (*e.g.* with minors), but how to enforce equality?

$$V_{=r} = V_{\leq r} \setminus V_{\leq r-1}$$

We would like to saturate, but how?

We use a local description:
For each r-minor A of M, we consider the ideal:

$$I(V_{\leq r}) + \langle U \cdot \det(A) - 1 \rangle \cap \mathbb{Q}[X, G]$$

which is generated by the $(r+1)$-minors of M containing A.

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Interlude 3: Cramer's rule and Schur complement

Cramer's rule / Gauß elimination (A is invertible)

$$\begin{array}{c c c} & \overset{k-1}{\longleftrightarrow} \quad \overset{1}{\longleftrightarrow} & \\ \begin{array}{r} k-1 \\ 1 \end{array} & \begin{bmatrix} A & B \\ C & d \end{bmatrix} & = \begin{bmatrix} I_{k-1} & 0 \\ \bullet & 1 \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & d - CA^{-1}B \end{bmatrix} \cdot \begin{bmatrix} I_{k-1} & \bullet \\ 0 & 1 \end{bmatrix} \end{array}$$

$$d - CA^{-1}B = \frac{\det(M)}{\det(A)}$$

Schur complement (A is invertible)

$$\begin{array}{c c c} & \overset{r}{\longleftrightarrow} \quad \overset{k-r}{\longleftrightarrow} & \\ \begin{array}{r} r \\ k-r \end{array} & \begin{bmatrix} A & B \\ C & D \end{bmatrix} & = \begin{bmatrix} I_r & 0 \\ \bullet & I_{k-r} \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \cdot \begin{bmatrix} I_r & \bullet \\ 0 & I_{k-r} \end{bmatrix} \end{array}$$

$$(i,j)\text{-entry of } D - CA^{-1}B = \frac{\det\left((r+1)\text{-minor “}A + \text{rowcol}\,(r+i, r+j)\text{”}\right)}{\det(A)}$$

Interlude 3: Cramer's rule and Schur complement

Cramer's rule / Gauß elimination (A is invertible)

$$
\begin{array}{cc}
 & \begin{array}{cc} \overset{\longleftrightarrow}{k-1} & \overset{\longleftrightarrow}{1} \end{array} \\
\begin{array}{c} k-1 \updownarrow \\ 1 \updownarrow \end{array} & \begin{bmatrix} A & B \\ C & d \end{bmatrix}
\end{array}
= \begin{bmatrix} I_{k-1} & 0 \\ \bullet & 1 \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & d - CA^{-1}B \end{bmatrix} \cdot \begin{bmatrix} I_{k-1} & \bullet \\ 0 & 1 \end{bmatrix}
$$

$$
d - CA^{-1}B = \frac{\det(M)}{\det(A)}
$$

Schur complement (A is invertible)

$$
\begin{array}{cc}
 & \begin{array}{cc} \overset{\longleftrightarrow}{r} & \overset{\longleftrightarrow}{k-r} \end{array} \\
\begin{array}{c} r \updownarrow \\ k-r \updownarrow \end{array} & \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\end{array}
= \begin{bmatrix} I_r & 0 \\ \bullet & I_{k-r} \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \cdot \begin{bmatrix} I_r & \bullet \\ 0 & I_{k-r} \end{bmatrix}
$$

$$
(i,j)\text{-entry of } D - CA^{-1}B = \frac{\det\left((r+1)\text{-minor ``} A + \text{rowcol}\,(r+i, r+j)\text{''}\right)}{\det(A)}
$$

Modelization using incidence varieties

Reminder: k = size of the matrix; r = target rank

Possible modelizations for determinantal varieties

- Minors: $\mathrm{rank}(A) \leq r \iff$ all $r+1$-minors of A are 0
- Incidence system: $\mathrm{rank}(A) \leq r \iff \exists\, L, A \cdot L = 0$ and $\mathrm{rank}(L) = k - r$

Minors:

- $\binom{k}{r+1}^2$ equations
- Codimension $(k-r)^2$

Incidence system:

- $k(k-r)$ new variables (entries of the matrix L)
- $(k-r)^2 + k(k-r)$ equations
- Codimension: $(k-r)^2 + k(k-r)$

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- Codimension: $(k-r)^2 + k(k-r)$

Properties of the incidence system (generically and in our situation)

- It forms a regular sequence (codimension = length) $\Longrightarrow$ equidimensional
- It defines a radical ideal

Consequence for the strategy

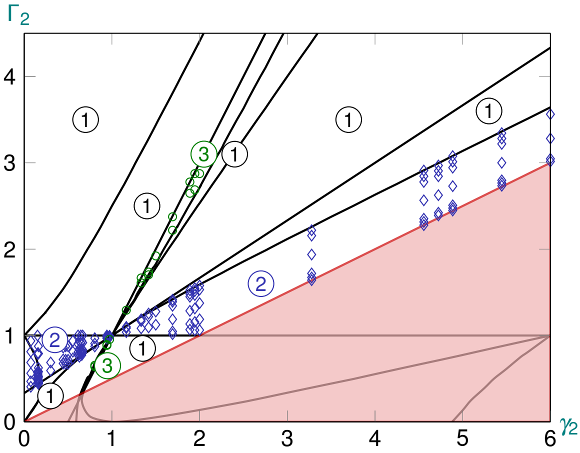
$K(\pi, V_{\leq r-1}(M))$ can be computed with the incidence system,
using maximal minors of the Jacobian matrix

Application to the contrast problem (benchmarks)

- Computations run on the matrix of the contrast optimization problem
  - Water: $\Gamma_1 = \gamma_1 = 1 \implies$ 2 parameters
  - General: $\gamma_1 = 1 \implies$ 3 parameters
- Results obtained with Maple
- Source code and full results available at `mercurey.gforge.inria.fr`

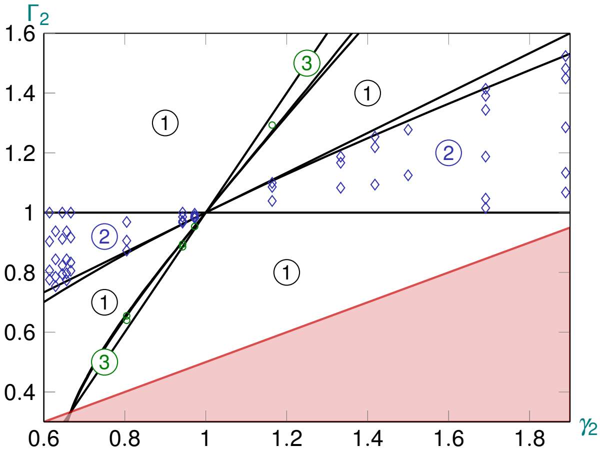
Elimination tool	Water (direct)	Water (det. strat.)	General (direct)	General (det. strat.)
Gröbner bases (`FGb`)	100 s	10 s	>24 h	46 × 200 s
Gröbner bases (`F5`)	-	1 s	-	110 s
Regular chains (`RegularChains`)	1050 s	-	>24 h	90 × 200 s

Results for the contrast problem in the case of water



Finishing the computations:

1. Classification algorithm $\to$ limits of the cells
2. Cylindrical algebraic decomposition $\to$ points in each cell
3. Gröbner basis computations for each point $\to$ count of singularities



Finishing the computations:

1. Classification algorithm $\rightarrow$ limits of the cells
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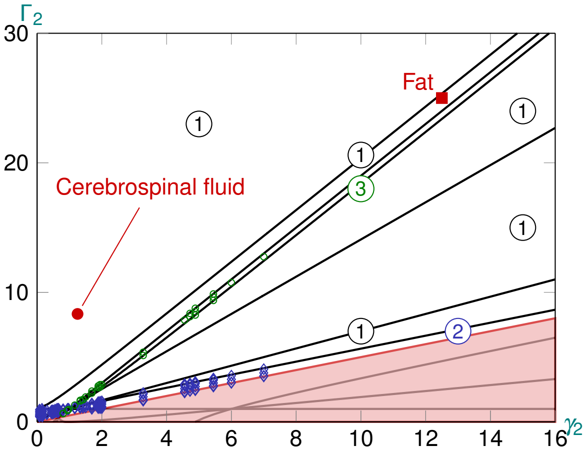
3. Gröbner basis computations for each point $\rightarrow$ count of singularities

Results for the contrast problem in the case of water (zoom out)



Finishing the computations:

1. Classification algorithm → limits of the cells
2. Cylindrical algebraic decomposition → points in each cell
3. Gröbner basis computations for each point → count of singularities

Conclusion and perspectives

This work

- Applications of algebraic methods to an optimal control problem
- Dedicated strategy for a classification problem related to one of the invariants

Perspectives

Algorithmically:

- Extension of the algorithms to structures of other invariants

And for the MRI problem:

- Direct relation between the invariants and properties of the trajectories?
- Is it possible to lift some approximations?
- Further studies, *e.g.* classification according to optimal contrast

The end?

Control problem

↓ Pontryagin

Description of extremal trajectories

↓

Classification of algebraic invariants of the singular dynamics

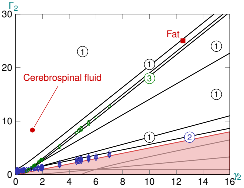
↓

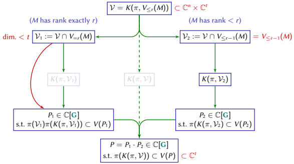
Classification of the trajectories

$$\dot{q} = F(\gamma, \Gamma, q) + uG(q)$$

- Bang arcs: $u \equiv \pm 1$
- Singular arcs: $u = \dfrac{D'}{D}$

$$\dot{X} = D\,F - D'\,G$$





Thank you!