On The Complexity Of Computing Gröbner Bases For Weighted Homogeneous Systems

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Polynomial system solving

Applications:
- Cryptography
- Physics, industry
- Mathematics...

Polynomial equations

$f_1(\mathbf{X}) = \cdots = f_m(\mathbf{X}) = 0$

Solutions,
e.g. find all the solutions
if finite (dimension 0)

- Numerical: give approximations of the solutions
  - Newton's method
  - Homotopy continuation method
- Symbolic: give exact solutions
  - Gröbner bases
  - Resultant method
  - Triangular sets
  - Geometric resolution

Computing Gröbner bases for generic systems: the normal strategy

Gröbner basis algorithms (e.g. F_5)

- Compute a basis by iteratively building and reducing matrices of polynomials of same degree
- Normal strategy: perform lowest-degree reductions first
- Degree = indicator of progress

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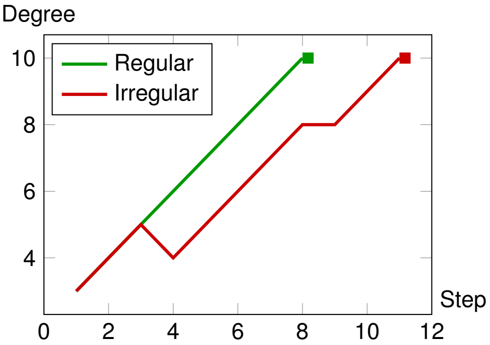

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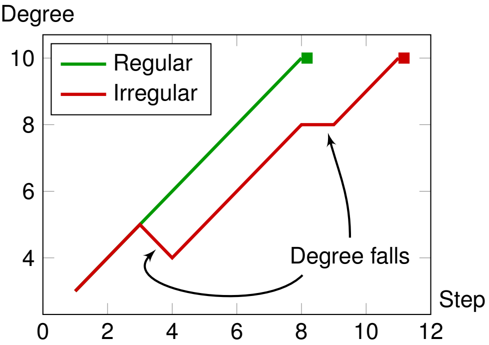

Degree fall?

- Definition: reduction resulting in a lower degree polynomial
- Example: $X \cdot (Y-1) - Y \cdot (X-1) = XY - YX + Y - X$
- Consequence: "next d" $< d+1$

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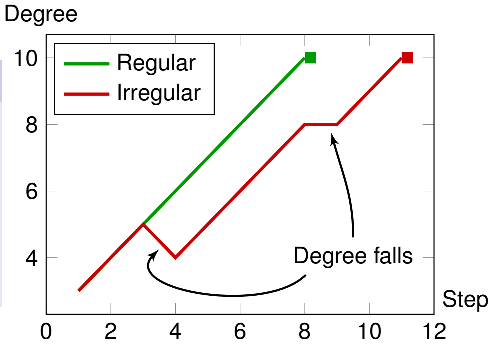

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Regular sequences $\implies$ algorithmic regularity!

- F_5-criterion: no reduction to zero in F_5 ($\iff$ all matrices have full-rank) for regular sequences
- Degree falls $\iff$ Reduction to zero of the highest degree components

$\rightsquigarrow$ Regularity in the affine sense = regularity of the highest degree components

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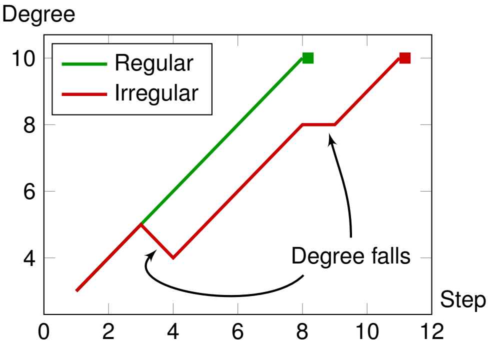

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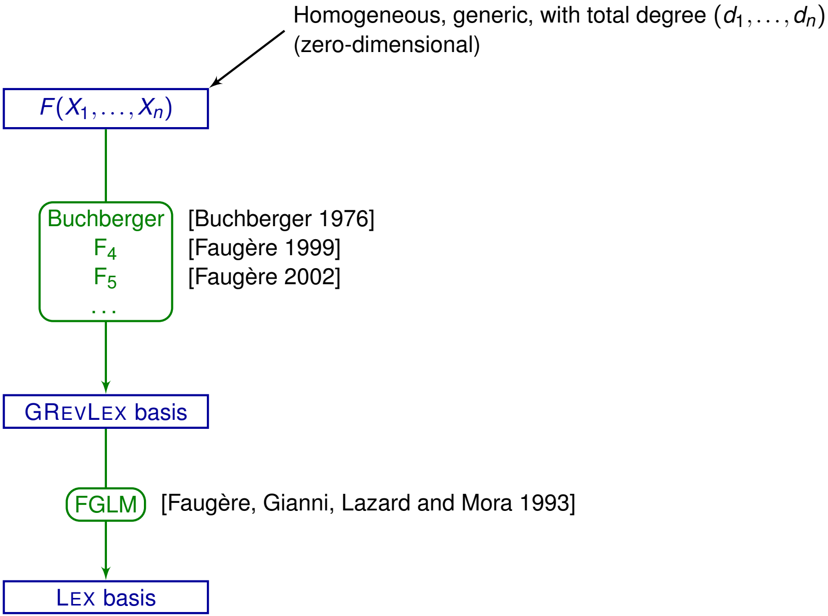
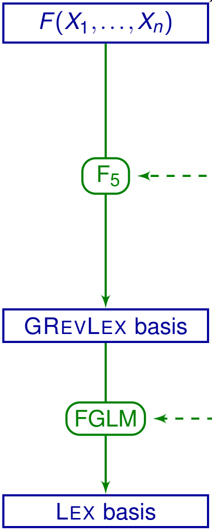
This notion depends on the homogeneous structure!

Strategy and complexity for generic homogeneous systems



Strategy and complexity for generic homogeneous systems

Homogeneous, generic, with total degree $(d_1, \ldots, d_n)$ (zero-dimensional)



Highest degree $\sim$ # of reduction steps

$$= d_{\text{reg}} \leq \sum_{i=1}^{n} (d_i - 1) + 1$$

Size of the matrix at degree $d = \binom{n+d-1}{d}$

Number of solutions $= \prod_{i=1}^{n} d_i$ (Bézout bound)

$$O\left(\binom{n + d_{\text{reg}} - 1}{d_{\text{reg}}}^3 + n \left(\prod_{i=1}^{n} d_i \right)^3 \right)$$

The weighted homogeneous structure: an example (1)

Discrete Logarithm Problem on Edwards elliptic curves (Faugère, Gaudry, Huot, Renault 2013)

$$0 = \begin{bmatrix}41518\\33900\\8840\\22855\\29081\end{bmatrix} X_5^{16} + \begin{bmatrix}49874\\32136\\34252\\24932\\11782\end{bmatrix} X_1^{8} + \begin{bmatrix}45709\\10698\\45336\\26076\\55993\end{bmatrix} X_1^{7}X_2 + \begin{bmatrix}46659\\59796\\38267\\39647\\27683\end{bmatrix} X_1^{6}X_2^{2} + \begin{bmatrix}32367\\23164\\64111\\63692\\29095\end{bmatrix} X_1^{5}X_2^{3} + \begin{bmatrix}37627\\25182\\59951\\60422\\11080\end{bmatrix} X_1^{4}X_2^{4} +$$

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Goal: compute a Gröbner basis

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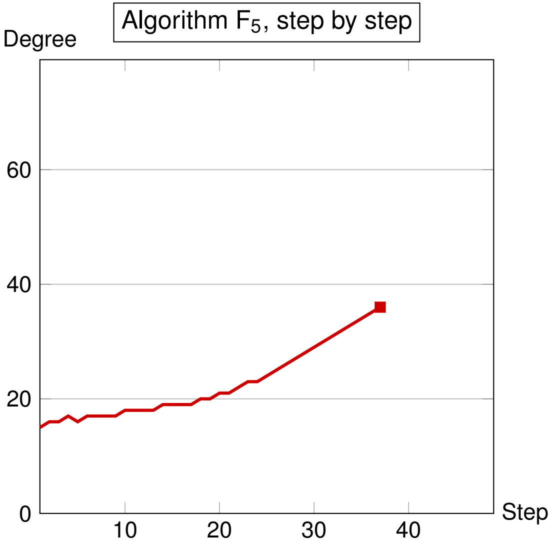

- Without weights:
  2 h (37 steps, $d_{reg} = 36$)
- With $W = (2, 2, 1, 1, 1)$:
  2 h (46 steps, $d_{reg} = 38$)
- With $W = (2, 2, 2, 2, 1)$:
  15 min (29 steps, $d_{reg} = 72$)

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Alt. strategy: use weights

$=$ substitute $X_i \leftarrow X_i^{w_i}$ for $W = (w_1, \ldots, w_5)$

What weights?

- $W = (1,1,1,1,1)$: nothing changed
- $W = (2,2,1,1,1)$: better...
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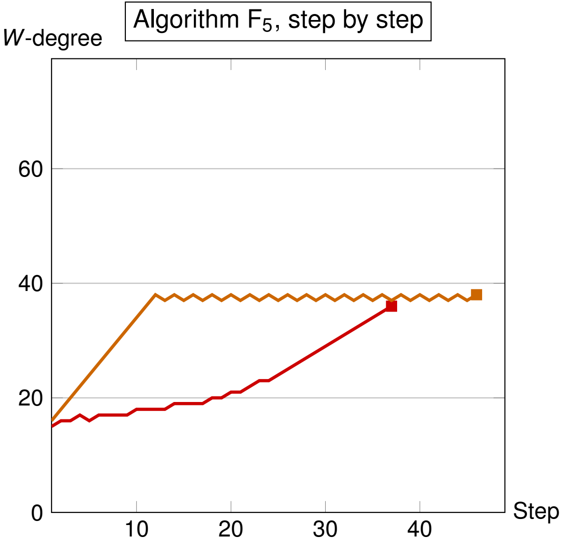

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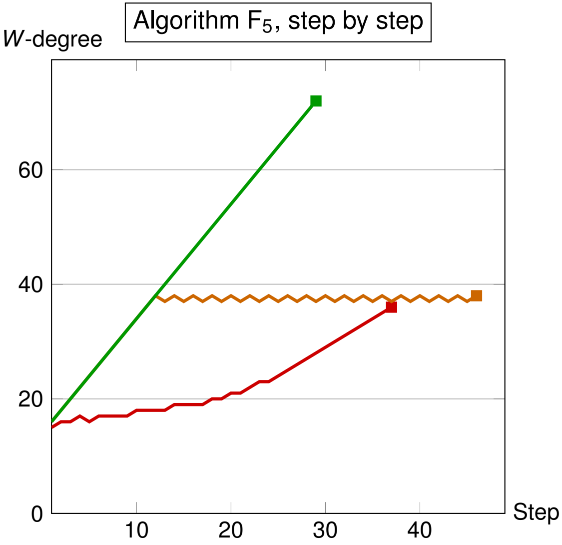

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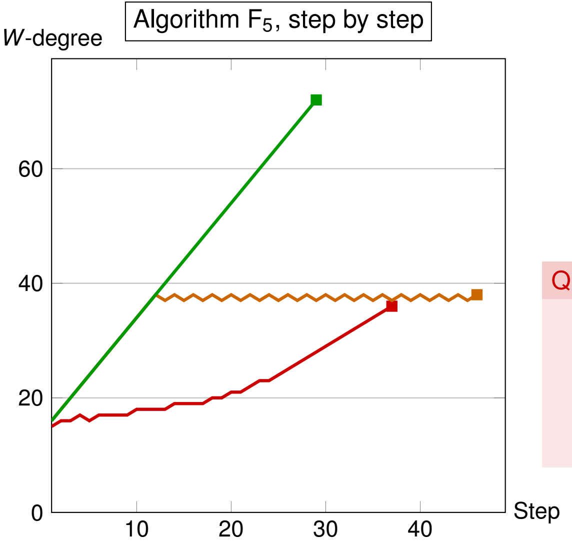

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Questions

- Explain the regularity?
- Complexity bounds?
- Why does FGLM become a bottleneck?

Weighted homogeneous systems: definitions

Definition (e.g. [Robbiano 1986], [Becker and Weispfenning 1993])

System of weights: $W = (w_1, \ldots, w_n) \in \mathbb{N}^n$

Weighted degree (or W-degree): $\deg_W(X_1^{\alpha_1} \ldots X_n^{\alpha_n}) = \sum_{i=1}^{n} w_i \alpha_i$

Weighted homogeneous polynomial: poly. containing only monomials of same W-degree

$\rightarrow$ Example: physical systems: Volume $=$ Area $\times$ Height

Weight 3 Weight 2 Weight 1

Given a general (non-weighted homogeneous) system and a system of weights

Computational strategy: weighted homogenize it as in the homogeneous case

Complexity estimates: consider the highest-W-degree components of the system

- Enough to study weighted homogeneous systems

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Main results: strategy and complexity results

$W = (w_1, \ldots, w_n)$

$F(X_1, \ldots, X_n), W$ ← W-Homogeneous, generic, with W-degree $(d_1, \ldots, d_n)$ (zero-dimensional)

↓

$F(X_1^{w_1}, \ldots, X_n^{w_n})$ ← Homogeneous, with total degree $(d_1, \ldots, d_n)$

↓

F$_5$ ← Highest W-degree

$$d_{W,\text{reg}} \leq \sum_{i=1}^{n} (d_i - 1) + 1 - \sum_{i=1}^{n} (w_i - 1) + w_n - 1$$

Size of the matrix at W-degree $d \simeq \dfrac{1}{\prod_{i=1}^{n} w_i} \dbinom{n+d-1}{d}$

↓

W-GRevLex basis of F

↓

FGLM ← Number of solutions $= \dfrac{\prod_{i=1}^{n} d_i}{\prod_{i=1}^{n} w_i}$ (weighted Bézout bound)

$$O\left(\left(\frac{1}{\prod_{i=1}^{n} w_i}\right)^3 \left[\binom{n + d_{W,\text{reg}} - 1}{d_{W,\text{reg}}}^3 + n\left(\prod_{i=1}^{n} d_i\right)^3\right]\right)$$

↓

Lex basis

Roadmap

Input

- $W = (w_1, \dots, w_n)$ system of weights
- $F = (f_1, \dots, f_m)$ generic sequence of W-homogeneous polynomials with W-degree $(d_1, \dots, d_m)$

General roadmap:

1. Find a generic property with "good" algorithmic and algebraic consequences
   - Regular sequences (dimension 0, $m = n$)
   - Noether position (positive dimension, $m \leq n$)
   - ...Semi-regular sequences (dimension 0, $m > n$)
2. Design new algorithms to take advantage of this structure
   - Adapt algorithms for the homogeneous case to the weighted homogeneous case
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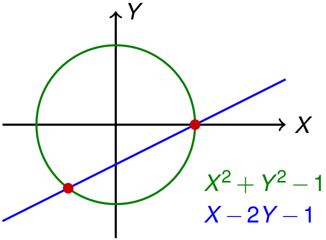
Regular sequences

Definition

$F = (f_1, \ldots, f_m)$ homo. $\in \mathbb{K}[\mathbf{X}]$ is regular iff

$$\begin{cases} \langle F \rangle \subsetneq \mathbb{K}[\mathbf{X}] \\ \forall i,\ f_i \text{ is no zero-divisor in } \mathbb{K}[\mathbf{X}]/\langle f_1, \ldots, f_{i-1} \rangle \end{cases}$$



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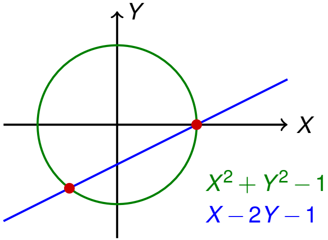
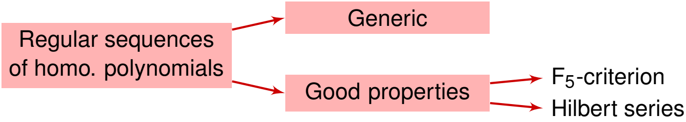
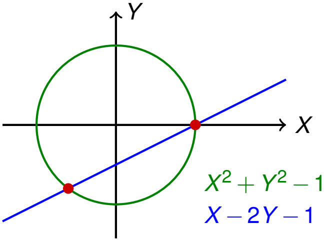




Definition

$F = (f_1, \ldots, f_m)$ weighted homo. $\in \mathbb{K}[\mathbf{X}]$ is regular iff

$$\begin{cases} \langle F \rangle \subsetneq \mathbb{K}[\mathbf{X}] \\ \forall i,\ f_i \text{ is no zero-divisor in } \mathbb{K}[\mathbf{X}]/\langle f_1, \ldots, f_{i-1} \rangle \end{cases}$$



Regular sequences of W-homo. polynomials → Generic if $\neq \varnothing$

Regular sequences of W-homo. polynomials → Good properties → F_5-criterion, Hilbert series

Properties of regular sequences

Hilbert series

$$\mathrm{HS}_{A/I}(T) = \sum_{d=0}^{\infty} (\text{rank defect of the } \mathrm{F}_5 \text{ matrix at degree } d) \cdot T^d$$

Properties

For regular sequences of homogeneous polynomials of degree d_i:

$$\mathrm{HS}_{A/I}(T) = \frac{(1-T^{d_1})\cdots(1-T^{d_m})}{(1-T)^n}$$

In zero dimension ($m = n$):

- Bézout bound on the degree: $D = \prod_{i=1}^{n} d_i$
- Macaulay bound on the degree of regularity: $d_{\mathrm{reg}} \leq \sum_{i=1}^{n} (d_i - 1) + 1$

Properties of regular sequences

Hilbert series

$$\mathrm{HS}_{A/I}(T) = \sum_{d=0}^{\infty} (\text{rank defect of the } \mathrm{F}_5 \text{ matrix at } W\text{-degree } d) \cdot T^d$$

Properties

For regular sequences of W-homogeneous polynomials of W-degree d_i:

$$\mathrm{HS}_{A/I}(T) = \frac{(1-T^{d_1})\cdots(1-T^{d_m})}{(1-T^{w_1})\cdots(1-T^{w_n})}$$

In zero dimension ($m = n$):

- Bézout bound on the degree: $D = \frac{\prod_{i=1}^{n} d_i}{\prod_{i=1}^{n} w_i}$
- Macaulay bound on the degree of regularity: $d_{\mathrm{reg}} \leq \sum_{i=1}^{n} (d_i - w_i) + \max\{w_j\}$

Limitations

Limitations of the regularity

- $m < n$ (positive dimension): no real information
- $m = n$ (zero dimension, complete intersection)
  - exact formula for d_{reg}?
  - d_{reg} depends on the order of the variables
  - Hilbert series: independent from that order
- $m > n$ (e.g. cryptography): no regular sequence

$\Longrightarrow$ Additional properties

- $m < n$: Noether position
- $m = n$: simultaneous Noether position
- $m > n$: semi-regular sequences

Noether position ($m < n$)

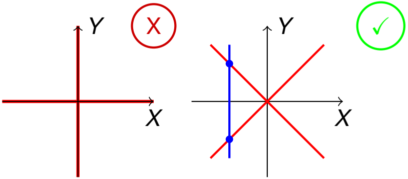
Definition

$F = (f_1, \ldots, f_m) \in \mathbb{K}[X_1, \ldots, X_n]$

is in Noether position iff

$(F, X_{m+1}, \ldots, X_n)$ is regular

"Regularity + selected variables"



Properties

- Generic if not empty
- True up to a generic change of coordinates if non-trivial changes exist

  (E.g. if $1 = w_n \mid w_{n-1} \mid \ldots \mid w_1$)
- Macaulay bound on d_{reg}: $d_{\mathrm{reg}} \leq \sum_{i=1}^{m} d_i - \sum_{i=1}^{m} w_i + \max_{1 \leq j \leq m} \{w_j\}$

  (only the first m weights matter)

Simultaneous Noether position ($m \leq n$)

Noether position = information on what variables are important
$\Rightarrow$ Good property for W-homogeneous systems in general

Definition

$F = (f_1, \dots, f_m) \in \mathbb{K}[X_1, \dots, X_n]$
is in simultaneous Noether position iff
$(f_1, \dots, f_j)$ is in Noether pos. for all j's

Properties

- $d_{\text{reg}} \leq \sum_{i=1}^{m} (d_i - w_i) + w_m$
- Better to have $w_m \leq w_j$ ($j \neq m$)

Order of the variables	w_m	d_{reg}	Macaulay's bound	New bound	F_5 time (s)
$X_1 > X_2 > X_3 > X_4$	1	210	229	210	101.9
$X_4 > X_3 > X_2 > X_1$	20	220	229	229	255.5

Generic W-homo. system, W-degree $(60, 60, 60, 60)$ w.r.t $W = (20, 5, 5, 1)$

Overdetermined case ($m > n$)

Equivalent definitions in the homogeneous case

$F = (f_1, \dots, f_m) \in \mathbb{K}[X_1, \dots, X_n]$ homogeneous is semi-regular

$\iff \forall k \in \{1, \dots, m\}, \forall d \in \mathbb{N}, (\cdot f_k) : (A/I_{k-1})_d \to (A/I_{k-1})_{d+d_k}$ is full-rank

$\iff \forall k \in \{1, \dots, m\}, \mathrm{HS}_{A/I_k} = \left\lfloor \frac{\prod_{i=1}^{k}(1 - T^{d_i})}{(1-T)^n} \right\rfloor_+$ (truncated at the first coef. ≤ 0)

Properties

- Conjectured to be generic (Fröberg)
- Proved in some cases (ex: $m = n + 1$)
- Practical and theoretical gains
- Asymptotic studies of d_{reg}

Overdetermined case ($m > n$)

Equivalent definitions in the weighted homogeneous case?

$F = (f_1, \ldots, f_m) \in \mathbb{K}[X_1, \ldots, X_n]$ W-homogeneous is semi-regular

$\overset{?}{\iff} \forall k \in \{1, \ldots, m\}, \forall d \in \mathbb{N}, (\cdot f_k) : (A/I_{k-1})_d \to (A/I_{k-1})_{d+d_k}$ is full-rank

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No equivalence without hypotheses on the weights

Ex: $n = 3$, $W = (3, 2, 1)$, $m = 8$, $D = (6, \ldots, 6)$:

$$\left\lfloor \frac{\prod_{i=1}^{m}(1 - T^{d_i})}{\prod_{i=1}^{n}(1 - T^{w_i})} \right\rfloor_+ = 1 + T + 2T^2 + 3T^3 + 4T^4 + 5T^5 - T^6 + 0T^7 - 6T^8 + \cdots$$

$$\mathrm{HS}_{A/I} = 1 + T + 2T^2 + 3T^3 + 4T^4 + 5T^5 + 0T^6 + T^7$$

Overdetermined case ($m > n$)

Equivalent definitions in the weighted homogeneous case

Assume that $1 = w_n \mid w_{n-1} \mid \dots \mid w_1$.

$F = (f_1, \dots, f_m) \in \mathbb{K}[X_1, \dots, X_n]$ W-homogeneous is semi-regular

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Roadmap

Input

- $W = (w_1, \dots, w_n)$ system of weights
- $F = (f_1, \dots, f_m)$ generic sequence of W-homogeneous polynomials with W-degree $(d_1, \dots, d_m)$

General roadmap:

1. Find a generic property with “good” algorithmic and algebraic consequences
   - Regular sequences (dimension 0, $m = n$)
   - Noether position (positive dimension, $m \leq n$)
   - ... Semi-regular sequences (dimension 0, $m > n$)
2. Design new algorithms to take advantage of this structure
   - Adapt algorithms for the homogeneous case to the weighted homogeneous case
3. Obtain complexity results for these algorithms

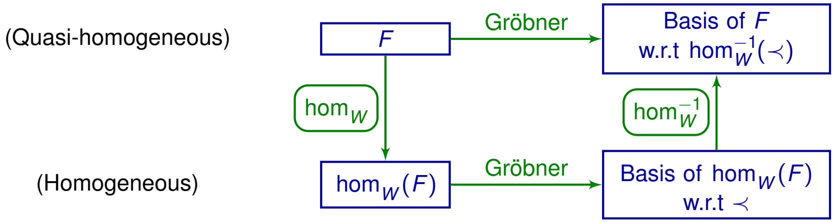
Algorithms: from weighted homogeneous to homogeneous

Transformation morphism

$$\begin{array}{rccc} \hom_W : & (\mathbb{K}[\mathbf{X}], W\text{-deg}) & \to & (\mathbb{K}[\mathbf{X}], \deg) \\ & f & \mapsto & f(X_1^{w_1}, \ldots, X_n^{w_n}) \end{array}$$

- Graded injective morphism
- Sends regular sequences on regular sequences
- $\text{S-Pol}(\hom_W(f), \hom_W(g)) = \hom_W(\text{S-Pol}(f, g))$

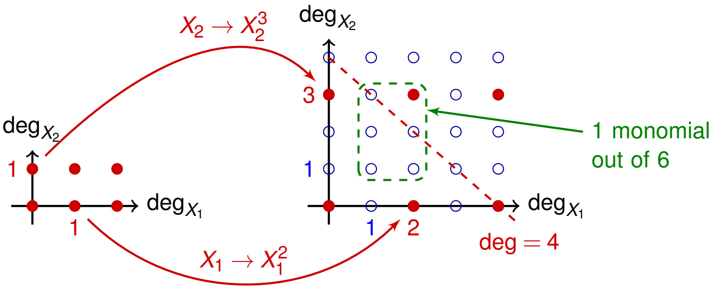
$\longrightarrow$ Good behavior w.r.t Gröbner bases



Size of the Macaulay matrices

Counting the monomials

- $\hom_W(F)$ lies in an algebra with a lot of useless monomials
- Count them: combinatorial object named Sylvester denumerants
- Result[1]: asymptotically $N_d \sim \dfrac{\#\text{Monomials of total degree } d}{\prod_{i=1}^n w_i}$

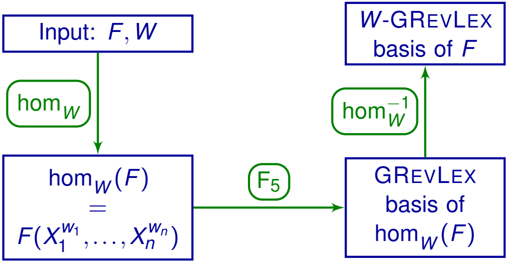


[1] Geir Agnarsson (2002). 'On the Sylvester denumerants for general restricted partitions'

Adapting the algorithms

Detailed strategy

- F_5 algorithm on the homogenized system
- FGLM algorithm on the weighted homogeneous system



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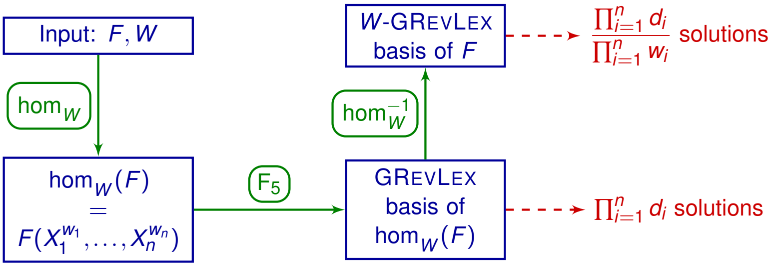

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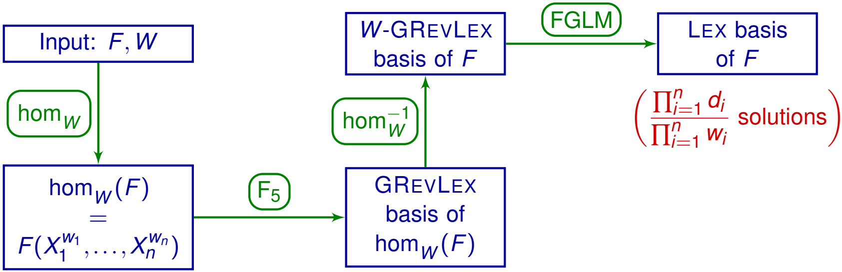

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Complexity

Input

- $W = (w_1, \dots, w_n)$
- $F = (f_1, \dots, f_n) \in \mathbb{K}[X_1, \dots, X_n]$ generic W-homogeneous

Complexity of F_5

$$\left(\frac{1}{\prod_{i=1}^n w_i}\right)^3 \binom{n + d_{\mathrm{reg}} - 1}{d_{\mathrm{reg}}}^3$$

- Asymptotic gain from the size of the matrices
- Practical gain from the weighted Macaulay bound (d_{reg})

Complexity of FGLM

$$\left(\frac{1}{\prod_{i=1}^n w_i}\right)^3 n \left(\prod_{i=1}^n d_i\right)^3$$

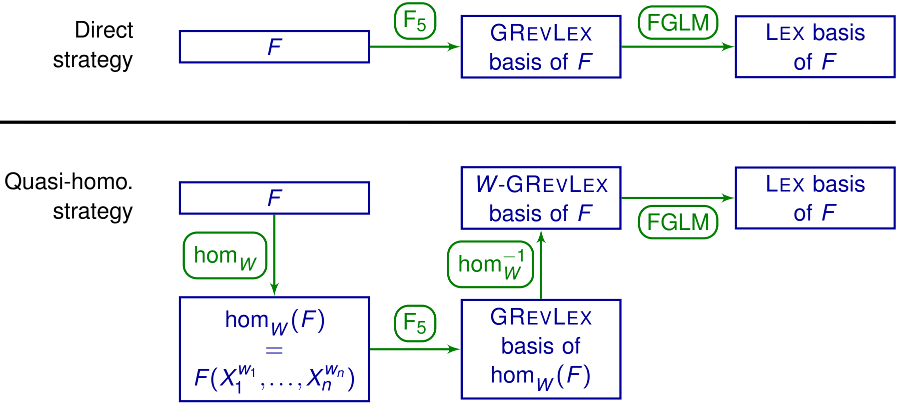
- Asymptotic gain from the weighted Bézout bound (number of solutions)

Benchmarking

F : affine system with a weighted homogeneous structure

$$f_i = \sum_\alpha c_\alpha m_\alpha \text{ with } \deg_W(m_\alpha) \leq d_i$$

Assumption: the highest W-degree components are regular (e.g. if F is generic)

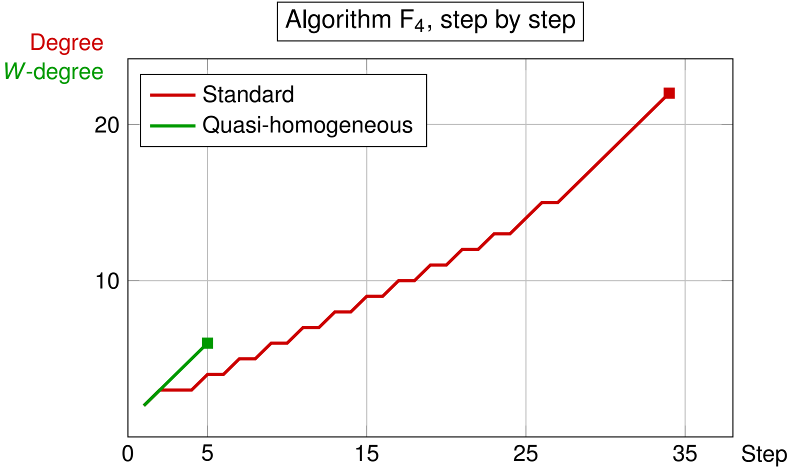


Experimental results

System	Normal (s)	Weighted (s)	Speed-up
DLP Edwards $n=5$, GREVLEX (F_5, FGb)	6461.2	935.4	6.9
DLP Edwards $n=5$, GREVLEX (F_4, Magma)	56195.0	6044.0	9.3
Invariant relations, Cyclic $n=5$, GREVLEX (F_4, Magma)	>75000	392.7	>191
Monomial relations, $n=26$, $m=52$, GREVLEX (F_4, Magma)	14630.6	0.2	73153
DLP Edwards $n=5$, LEX (Sparse-FGLM, FGb)	6835.6	2164.4	3.2
Invariant relations, Cyclic $n=5$, ELIM (F_4, Magma)	NA	382.5	NA
Monomial relations, $n=26$, $m=52$, ELIM (F_4, Magma)	17599.5	8054.2	2.2

A run of F_4 on an inversion example

Ideal of relations between 50 monomials of degree 2 in 25 variables



- 50 equations of (W-)degree 2 in 75 variables
- GRevLex ordering (e.g. for a 2-step strategy)
- Without weights: 3.9 h (34 steps reaching degree 22)
- With weights: 0.1 s (5 steps reaching W-degree 6)

Conclusion

What we have done

- Theoretical results for weighted homogeneous systems under generic assumptions
- Computational strategy for weighted homogeneous systems
- Complexity results for F_5 and FGLM for this strategy
  - Bound on the maximal degree reached by the F_5 algorithm
  - Complexity overall divided by $(\prod w_i)^3$

Consequences

- Successfully applied to a cryptographical problem
- Wide range of potential applications

Perspectives

- Affine systems: find the most appropriate system of weights
- Additional structure: weighted homo. for several systems of weights, weights ≤ 0...

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Thank you for your attention!