Determinantal sets, singularities and application to optimal control in medical imagery

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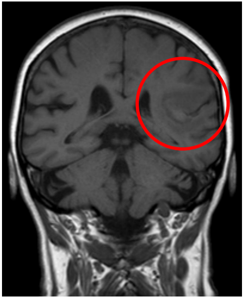
Physical problem

(N)MRI = (Nuclear) Magnetic Resonance Imagery

1. Apply a magnetic field to a body
2. Measure the radio waves emitted in reaction

Goal = optimize the contrast = distinguish two biological matters from this measure



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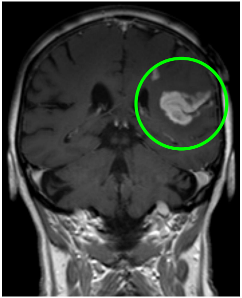
Bad contrast (no enhancement)



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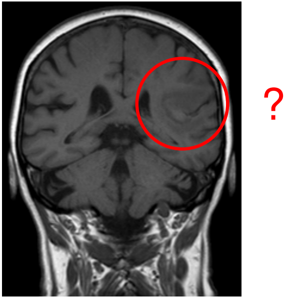

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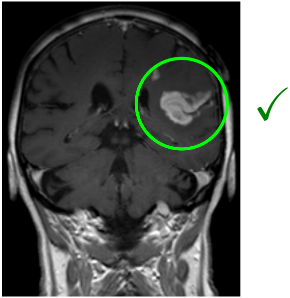

Good contrast (enhanced)

Known methods:

- inject contrast agents to the patient: potentially toxic
- make the field variable to exploit differences in relaxation times

  $\Longrightarrow$ requires finding optimal settings depending on the relaxation parameters

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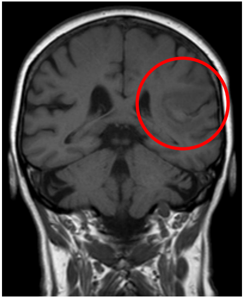

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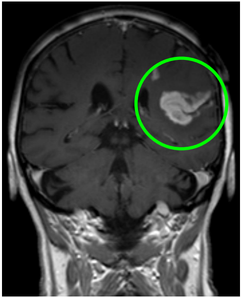

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Examples of relaxation parameters:

- Water: $\gamma = \Gamma = 0.01\,\text{Hz}$
- Cerebrospinal fluid: $\gamma = 0.02\,\text{Hz}$, $\Gamma = 0.10\,\text{Hz}$
- Fat: $\gamma = 0.15\,\text{Hz}$, $\Gamma = 0.31\,\text{Hz}$

Numerical approach and computational problem

The Bloch equations

$$\begin{cases} \dot{y}_i &= -\Gamma_i y_i - u z_i \\ \dot{z}_i &= -\gamma_i (1 - z_i) + u y_i \end{cases}$$

$(i = 1, 2)$

Saturation method

Find a path u so that after some time T:

- matter 1 saturated: $y_1(T) = z_1(T) = 0$
- matter 2 "maximized": $|(y_2(T), z_2(T))|$ maximal

Glaser's team, 2012 : method from Optimal Control Theory

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Problem: analyze the behavior of the control through algebraic invariants

- Example: singular feedback control: $u = \dfrac{D'}{D}$ (D, D' polynomials in y,z,γ,Γ)
- Geometry of $\{D = 0\}$?
- Study of the singular points of $\{D = 0\}$ for each value of $\gamma_1, \Gamma_1, \gamma_2, \Gamma_2$
- Examples with water: [Bonnard, Chyba, Jacquemard, Marriott, 2013]
  - Water/Fat : 1 point
  - Water/Cerebrospinal fluid : 1 point

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Questions

- Is there always 1 singular point for pairs involving water?
- If not, how many possible families of parameters can we separate?

Statement of the semi-algebraic problem

The D invariant: equations of a determinantal system

- $$M := \begin{pmatrix} -\Gamma_1 y_1 & -z_1 - 1 & -\Gamma_1 + (\gamma_1 - \Gamma_1) z_1 & (2\gamma_1 - 2\Gamma_1) y_1 \\ -\gamma_1 z_1 & y_1 & (\gamma_1 - \Gamma_1) y_1 & 2\Gamma_1 - \gamma_1 - (2\gamma_1 - 2\Gamma_1) z_1 \\ -\Gamma_2 y_2 & -z_2 - 1 & -\Gamma_2 + (\gamma_2 - \Gamma_2) z_2 & (2\gamma_2 - 2\Gamma_2) y_2 \\ -\gamma_2 z_2 & y_2 & (\gamma_2 - \Gamma_2) y_2 & 2\Gamma_2 - \gamma_2 - (2\gamma_2 - 2\Gamma_2) z_2 \end{pmatrix}$$
- $D := \text{determinant}(M)$
- $$V := \left\{ D = \frac{\partial D}{\partial y_1} = \frac{\partial D}{\partial z_1} = \frac{\partial D}{\partial y_2} = \frac{\partial D}{\partial z_2} = 0 \right\}$$

The Bloch ball: inequalities

- $$B := \left\{ \begin{array}{l} {y_1}^2 + (z_1 + 1)^2 \leq 1 \\ {y_2}^2 + (z_2 + 1)^2 \leq 1 \end{array} \right\}$$

Goal

Classification of the real fibers of the projection of $V \cap B$ onto the parameter space

State of the art and contributions

State of the art:

- General tool: Cylindrical Algebraic Decomposition
  [Collins, 1975]
- Specific tools for roots classification
  [Yang, Hou, Xia, 2001]
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Problem

- None of these algorithms can solve the problem efficiently:
  - 1050 s in the case of water ($\gamma_1 = \Gamma_1 = 1 \to 2$ parameters)
  - > 24 h in the general case (3 parameters)
- Can we exploit the determinantal structure to go further?

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Main results

- Dedicated strategy for real roots classification for determinantal systems
- Can use existing tools for elimination
- Main refinements:
  - Rank stratification
  - Incidence varieties

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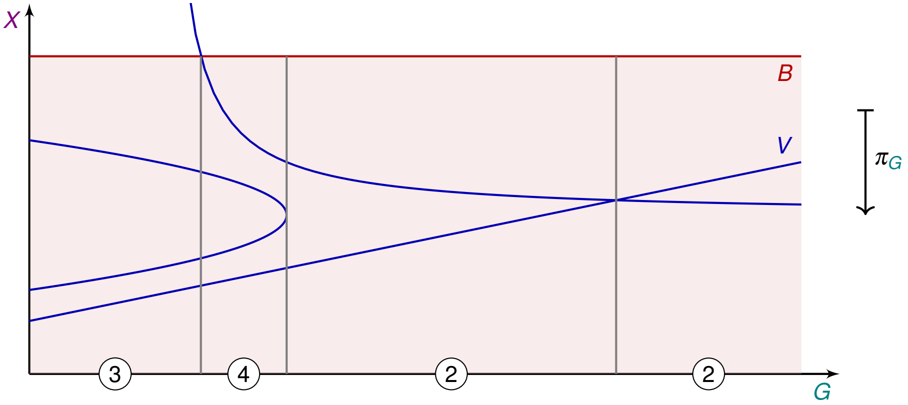
Main results

- Dedicated strategy for real roots classification for determinantal systems
- Can use existing tools for elimination
- Main refinements:
  - Rank stratification
  - Incidence varieties

- Faster than general algorithms:
  - 10 s in the case of water
  - 4 h in the general case
- Results for the application
  - Full classification
  - Answers to the experimental questions for water: there can be 1, 2 or 3 singularities

Classification strategy



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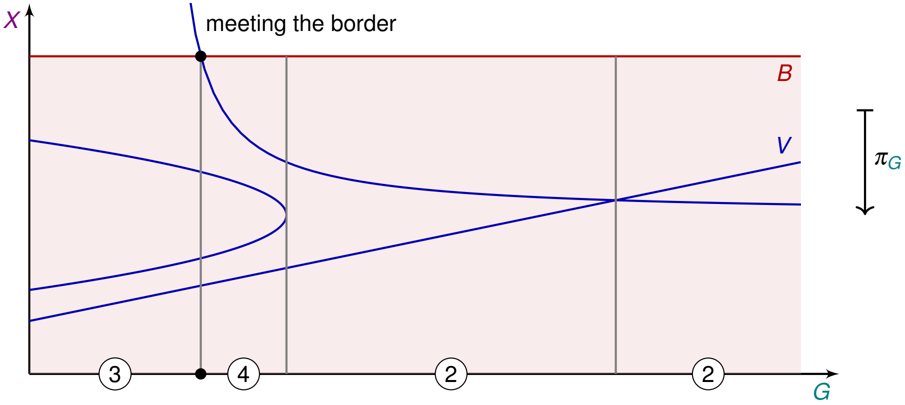

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- projections of points where V meets the border of the semi-algebraic domain

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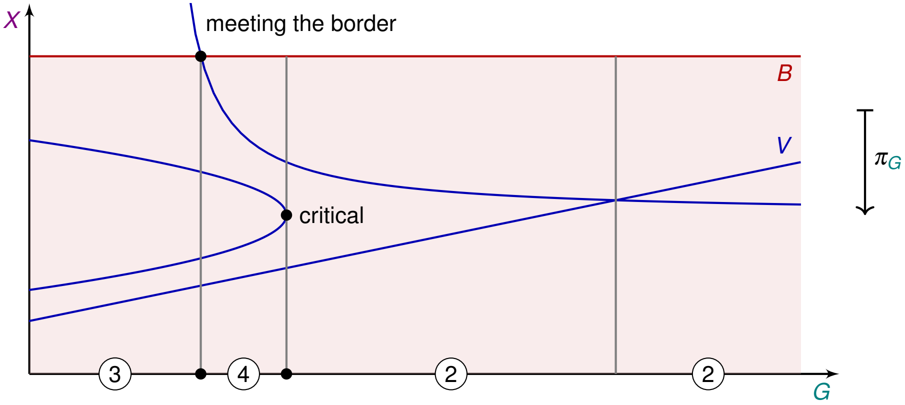

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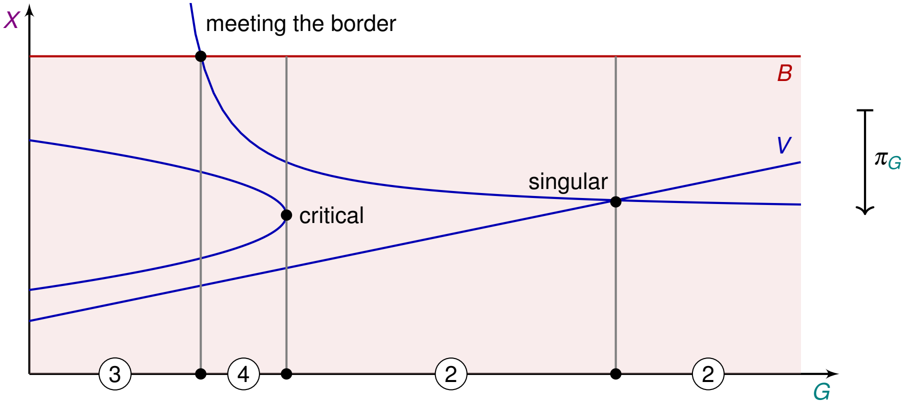

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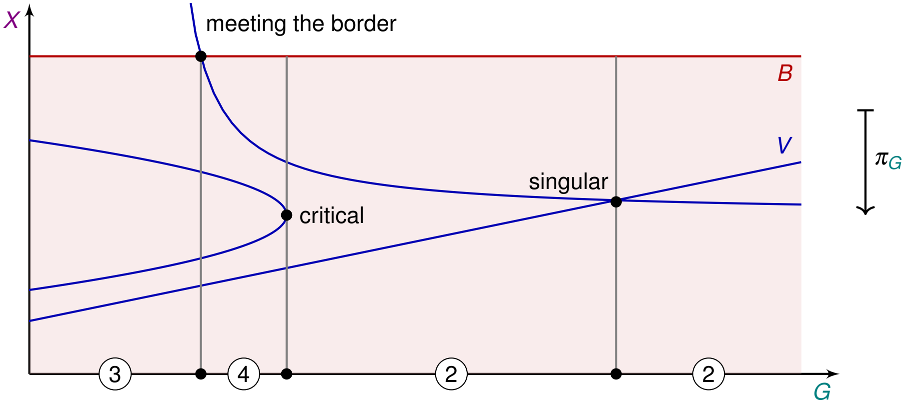

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We want to compute $P \in \mathbb{Q}[G]$ with $P \neq 0$ and P vanishing at all these points

How to compute these points

Goal: compute $P \in \mathbb{Q}[\gamma, \Gamma]$ such that

$$V(P) \supset \pi((V \cap \partial B) \cup \mathrm{Sing}(V) \cup \mathrm{Crit}(\pi, V))$$

with

- π : projection onto the parameters γ_i, Γ_i
- $V = \left\{ D = \frac{\partial D}{\partial y_1} = \frac{\partial D}{\partial z_1} = \frac{\partial D}{\partial y_2} = \frac{\partial D}{\partial z_2} = 0 \right\}$
- $B = \left\{ {y_1}^2 + (z_1 + 1)^2 \leq 1, {y_2}^2 + (z_2 + 1)^2 \leq 1 \right\}$

Intersection with the border

Compute:

- $V \cap \{{y_1}^2 + (z_1 + 1)^2 = 1\}$
- $V \cap \{{y_2}^2 + (z_2 + 1)^2 = 1\}$

and their image through π

(polynomial elimination)

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Critical and singular points

$$(\mathbf{y}, \mathbf{z}, \gamma, \Gamma) \in \mathrm{Sing}(V) \cup \mathrm{Crit}(\pi, V)$$
$$\iff \mathrm{Jac}(F, (\mathbf{y}, \mathbf{z})) \text{ has rank } < d$$

Requirements

- F generates the ideal of $V \implies$ radical
- V is equidimensional with codimension d

Properties of determinantal systems

Determinantal systems

- $A = k \times k$-matrix filled with polynomials in n variables $\mathbf{X}$ and t parameters $\mathbf{G}$
- $1 \leq r < k$ target rank
- Determinantal variety: $V_{\leq r}(A) = \{(\mathbf{x}, \mathbf{g}) : \mathrm{rank}(A(\mathbf{x}, \mathbf{g})) \leq r\}$

Our system: $n = 4$, $k = 4$, $r = 3$, $V = \mathrm{Sing}_{y_1, y_2, z_1, z_2}(V_{\leq r}(M))$

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With a generic matrix A

- $\mathrm{Sing}_{\mathbf{X}}(V_{\leq r}(A)) = V_{\leq r-1}(A)$

- $V_{\leq r}(A)$ equidimensional
- $\dim(V_{\leq r}(A)) = n + t - (k - r)^2$

  $\implies \dim(V_{\leq r-1}(A)) = t$

With our specific matrix M

- $V_{\leq r-1}(M) \subset V$
- $\dim(V) = t$
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Rank stratification

Goal: $P \subset \mathbb{Q}[\gamma, \Gamma]$ such that $V(P) \supset \pi(\mathrm{Sing}(V) \cup \mathrm{Crit}(\pi, V))$ with $\pi : \mathbb{C}^{n+t} \to \mathbb{C}^t$

Rank stratification

$$V = (V \cap V_{=r}) \cup (V \cap V_{\leq r-1})$$

- $V \cap V_{=r}$: dimension $< t$
- $V \cap V_{\leq r-1} = V_{\leq r-1}$: t-equidimensional

Consequence for the strategy

1. Compute P_1 such that $V(P_1) \supset \pi(V \cap V_{=r})$
2. Compute P_2 such that $V(P_2) \supset \pi(\mathrm{Sing}(V_{\leq r-1}(M)) \cup \mathrm{Crit}(\pi, V_{\leq r-1}(M)))$
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Change of model: incidence varieties

Reminder: $k =$ size of the matrix; $r =$ target rank

Possible modelizations for determinantal varieties

- Minors: $\mathrm{rank}(A) \leq r \iff$ all $r+1$-minors of A are 0
- Incidence system: $\mathrm{rank}(A) \leq r \iff \exists L, A \cdot L = 0$ and $\mathrm{rank}(L) = k - r$

Minors:

- $\binom{k}{r+1}^2$ equations
- Codimension $(k-r)^2$

Incidence system:

- $k(k-r)$ new variables (entries of the matrix L)
- $(k-r)^2 + k(k-r)$ equations
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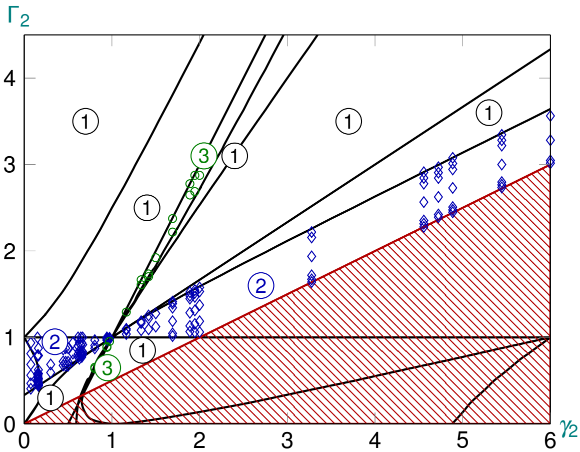
Properties of the incidence system (generically and in our situation)

- It defines a radical ideal
- It forms a regular sequence (codimension = length)
  $\Longrightarrow$ Critical points characterized by maximal minors of the Jacobian matrix

Consequence for the strategy

$\mathrm{Sing}(V_{\leq r-1}(M)) \cup \mathrm{Crit}(\pi, V_{\leq r-1}(M))$ can be computed with the incidence system

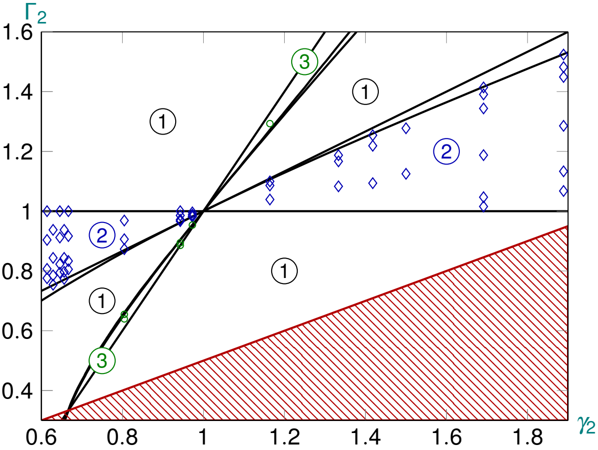
Results for water



Answers to the questions

- There can be 1, 2 or 3 singular points in the fibers
- We can separate 3 families of biological matters

Results for water (zoom in)



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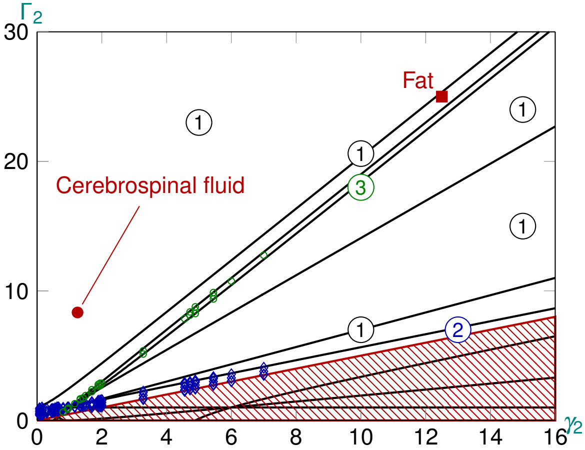

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Timings

- Computations run on the matrix of the contrast optimization problem
  - Water: $\Gamma_1 = \gamma_1 = 1 \implies$ 2 parameters
  - General: $\gamma_1 = 1 \implies$ 3 parameters
- Results obtained with Maple
- Source code and full results available at `mercurey.gforge.inria.fr`

Elimination tool	Water (direct)	Water (det. strat.)	General (direct)	General (det. strat.)
Gröbner bases (`FGb`)	100 s	10 s	$>$24 h	46×200 s
Gröbner bases (`F5`)	-	1 s	-	110 s
Regular chains (`RegularChains`)	1050 s	-	$>$24 h	90×200 s

Conclusion and perspectives

What has been done?

- Algorithmic strategy for real roots classification with determinantal varieties
- Exploiting the determinantal structure
- Successfully applied in an application to contrast optimization in MRI

The future

- Complexity bounds
- Contrast optimization: other criteria of the classification still need to be studied
- New numerical questions about small areas in the classification

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