Algebraic classification related to contrast optimization for MRI

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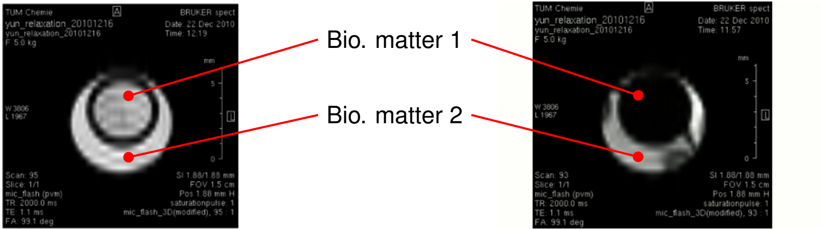
Physical problem

(N)MRI = (Nuclear) Magnetic Resonance Imagery

1. apply a magnetic field to a body
2. measure the radio waves emitted in reaction

Optimize the contrast = be able to distinguish two biological matters from this measure



Bad contrast (no enhancement) — Good contrast (enhanced)

Known methods:

- inject contrast agents to the patient: potentially toxic
- make the field variable to exploit differences in relaxation times
  $\Longrightarrow$ requires finding optimal settings depending on the relaxation parameters

(Images: Pr. Steffen Glaser, Tech. Univ. München)

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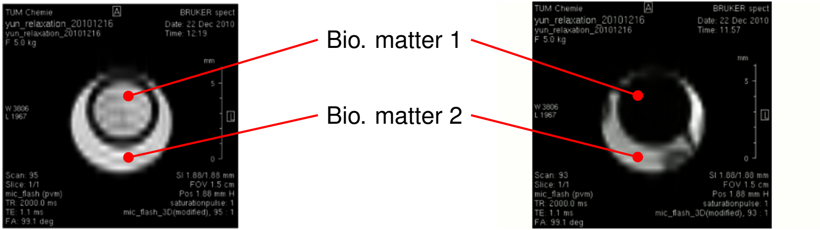

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Examples:

- Bio. matter 1: Deoxygenated blood ($\gamma_1 \simeq 0.74\,\text{Hz}$, $\Gamma_1 = 20\,\text{Hz}$)
- Bio. matter 2: Oxygenated blood ($\gamma_2 = 0.74\,\text{Hz}$, $\Gamma_2 \simeq 5\,\text{Hz}$)

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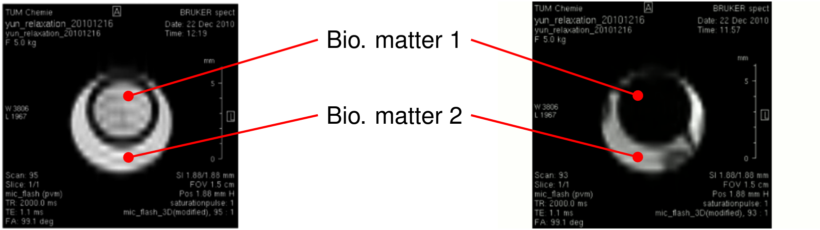

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Examples:

- Bio. matter 1: Water ($\gamma_1 = \Gamma_1 = 0.4\,\text{Hz}$)
- Bio. matter 2: Cerebrospinal fluid ($\gamma_2 = 0.5\,\text{Hz}$, $\Gamma_2 \simeq 3.3\,\text{Hz}$)

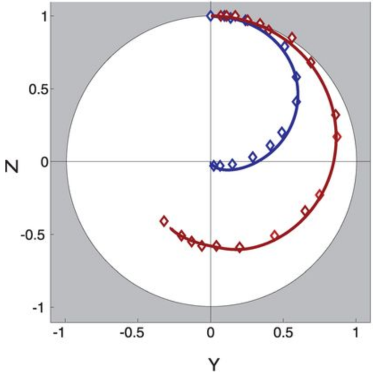
(Images: Pr. Steffen Glaser, Tech. Univ. München)

Numerical approach

The Bloch equations

$$\begin{cases} \dot{y}_i &= -\Gamma_i y_i - u \cdot z_i \\ \dot{z}_i &= -\gamma_i(1 - z_i) + u \cdot y_i \end{cases} \quad (i = 1, 2)$$

Bonnard, Glaser : Control Theory problem



Saturation method

Find a path $t \mapsto u(t)$ so that after some time T:

- matter 1 saturated: $y_1(T) = z_1(T) = 0$
- matter 2 “maximized”: $|(y_2(T), z_2(T))|$ maximal

Problem (Bonnard et al. 2013)

- The length T of the path is not bounded
- Goal: better understand the control problem to obtain optimal solutions
- Classify invariants of a related differential equation

(Images: Pr. Steffen Glaser, Tech. Univ. München)

The invariants D and D'

Expression for D (D' has an analogous definition)

- $M := \begin{pmatrix} -\Gamma_1 y_1 & -z_1 - 1 & -\Gamma_1 + (\gamma_1 - \Gamma_1)\, z_1 & (2\,\gamma_1 - 2\,\Gamma_1)\, y_1 \\ -\gamma_1 z_1 & y_1 & (\gamma_1 - \Gamma_1)\, y_1 & 2\,\Gamma_1 - \gamma_1 - (2\,\gamma_1 - 2\,\Gamma_1)\, z_1 \\ -\Gamma_2 y_2 & -z_2 - 1 & -\Gamma_2 + (\gamma_2 - \Gamma_2)\, z_2 & (2\,\gamma_2 - 2\,\Gamma_2)\, y_2 \\ -\gamma_2 z_2 & y_2 & (\gamma_2 - \Gamma_2)\, y_2 & 2\,\Gamma_2 - \gamma_2 - (2\,\gamma_2 - 2\,\Gamma_2)\, z_2 \end{pmatrix}$
- $D := \det(M)$

Properties:

- Homogeneous in the parameters γ_i, Γ_i, degree 3
- Degree 4 in the variables y_i, z_i

Now the problem is algebraic!

Goal

Classification of the invariants in terms of Γ_i, γ_i:

- Singularities of $\{D = 0\}$
- Surface $\{D = D' = 0\}$
- Curve of singularities of $\{D = D' = 0\}$

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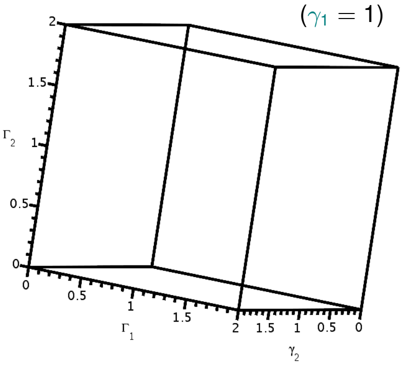


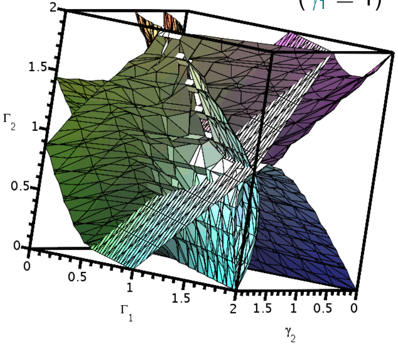
Singularities are invariant under $(y_i \mapsto -y_i)$

Classification in terms of Γ_i, γ_i:

- Generically: 4 pairs of singularities for each value of the parameters
- 3 pairs for each value on a surface with 5 components
  - one hyperplane
  - one quadric
  - one quartic
  - one degree 14 surface
  - one degree 24 surface
- 2 pairs for each value on a curve with many components
- 1 pair for each value on a set of points. . .

Overview of the results: singularities of $\{D = 0\}$ in terms of Γ_i, γ_i

$(\gamma_1 = 1)$



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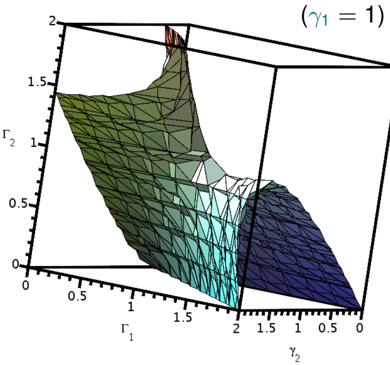

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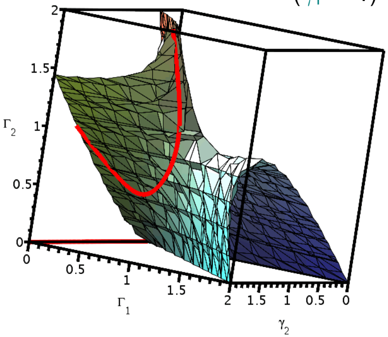

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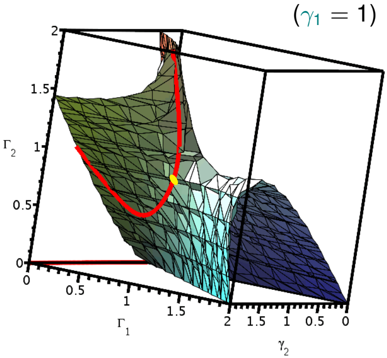

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Tool: Gröbner bases

What is it?

- Tool for solving polynomial systems
- If finite number of (complex) solutions: enumerations of the solutions as:

$$\begin{cases} P_1(X_1) = 0 \\ X_i - P_i(X_1) = 0 \end{cases}$$

- For systems with positive dimension: allows to compute projections
- Known since the 60s, now available in most computer algebra software

Advantages

- Exact computations: no solutions are left out
- Able to take advantage of algebraic or geometric structures
- More equations is usually better!

Caveats

- Long computations, complexity not known beforehand
- Complicated results (high degree, large polynomials)
- Global method: we can only localise on dense subsets

How do we use Gröbner bases on this problem?

Key idea: importance of the modelization

- The complexity depends on the system, rather than on its solutions
- Idea: choose a particular system with nicer properties
  - Examples: lower degree, less indeterminates, more equations...
- Usually, it means a tradeoff!

State of the art for the current problem

- General case $\longrightarrow$ 4 variables, 4 parameters — Intractable
- Particular cases $\longrightarrow$ 0 parameters — (Bonnard et al. 2013)

Application to the current problem

Filling the gap between the two extremes above

- Simplification by homogeneity $\gamma_1 = 1 \longrightarrow$ 3 parameters — Intractable
- Intermediate cases (e.g. water: $\gamma_1 = \Gamma_1 = 1$) $\longrightarrow$ 2 parameters — Solved

Attacking the general classification: decompositions into subproblems

Example of decomposition: rank of the matrix

We split the problem depending on the rank of the matrix:

$$\det(M) = 0 \implies \begin{cases} \text{rank}(M) = 3 \\ \quad \text{or} \\ \text{rank}(M) < 3 \end{cases}$$

Why the rank? Because of...

Theorem

Consider

$$M = (P_{i,j}(\mathbf{X}))_{1 \le i,j \le n}$$

Then generically:

$$\det(M) \text{ singular} \implies \text{rank}(M) < n - 1$$

With this specific matrix, the theorem does not apply.

$\implies$ There are solutions in both branches.

Case	Solutions in $\Gamma_1, \Gamma_2, \gamma_2$
$\text{rank}(M) < 3$	Dimension 3
$\text{rank}(M) = 3$	Dimension 2

Classification in the generic case rank$(M) < 3$

We append all 3×3 minors of M to the system

(Remember: more equations is better!)

Through a (long) Gröbner basis computation, we can find in the ideal:

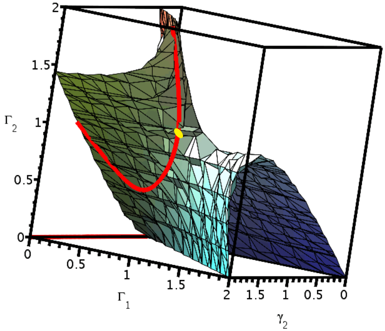
$$P = \sum_{d=0}^{4} a_d(\Gamma_1, \Gamma_2, \gamma_2) {y_2}^{2d}$$

It is a large polynomial (1776 monomials...) but with a nice structure:

- Degree 4 in y_2^2
- Non-irreducible coefficients in y_2, high degree common factors

Classification: number of roots of $P(y_2)$

- Generically: 4 pairs of opposite solutions
- If $a_4 = 0$ or disc$(P) = 0$, generically: 3 pairs of solutions
  - 3 components from the factorization of a_4
  - 2 components from the factorization of disc(P)
- ...



Example of change of model: what to do if $\mathrm{rank}(M) = 3$?

Theorem

Consider $M = (P_{i,j}(\mathbf{X}))_{1 \le i,j \le n}$ and let $\mathcal{I}$ be the incidence variety defined by

$$\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

If $(\mathbf{x})$ is a point of $\{\det(M(\mathbf{x})) = 0\}$, then:

- there exists a non-zero vector $\Lambda = (\lambda_i)$ such that $(\mathbf{x}, \Lambda) \in \mathcal{I}$

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If $(\mathbf{x})$ is a singular point of $\{\det(M(\mathbf{x})) = 0\}$ such that $M(\mathbf{x})$ has rank $n-1$, then:

- there exists a non-zero vector $\Lambda = (\lambda_i)$ such that $(\mathbf{x}, \Lambda) \in \mathcal{I}$, and
- Λ is unique up to scalar multiplication, and
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$$\left\langle D, \frac{\partial D}{\partial y_i}, \frac{\partial D}{\partial z_i}, \boxed{M \cdot \Lambda, \operatorname{rank}\left(\nabla_{y_i, z_i}(M \cdot \Lambda) < 4\right)}, \boxed{\mathcal{M}_k \neq 0}, \boxed{\lambda_i = 1} \right\rangle_{\substack{1 \le k \le 16 \\ 1 \le i \le 4}}$$

Singular point of the incidence variety

Non-zero 3×3 minor (16 choices)

Non-zero coordinate (4 choices)

Conclusion and perspectives

What has been done?

Part of the classification of invariants for the saturation problem

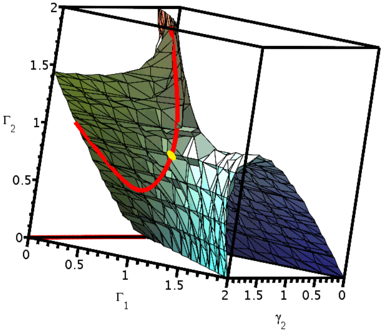
- Exhaustive classification in some particular cases (water)
- Some branches entirely explored in full generality

Still work in progress

- Some branches not solved yet in full generality
- Some parameters of the classification still need to be studied (D')

Applications

- New control policies for contrast optimisation for the MRI
- More generally, computational strategy applicable to similar problems



Thank you for your attention!

Merci pour votre attention!