# On The Complexity Of Computing Gröbner Bases For Weighted Homogeneous Systems


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# An example

Finding the relations between 50 monomials of degree 2 in 25 variables

| | | | | |
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| $X_{15}X_{25} - T_1$ | $X_{21}X_{22} - T_{11}$ | $X_{12}X_{14} - T_{21}$ | $X_3X_{21} - T_{31}$ | $X_1X_{12} - T_{41}$ |
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| $X_8X_{15} - T_3$ | $X_{12}^2 - T_{13}$ | $X_6X_8 - T_{23}$ | $X_{15}X_{25} - T_{33}$ | $X_{16}X_{19} - T_{43}$ |
| $X_6X_8 - T_4$ | $X_1X_2 - T_{14}$ | $X_{14}^2 - T_{24}$ | $X_{13}X_{20} - T_{34}$ | $X_{12}X_{18} - T_{44}$ |
| $X_5X_{12} - T_5$ | $X_{25}^2 - T_{15}$ | $X_8X_{21} - T_{25}$ | $X_6X_{19} - T_{35}$ | $X_7X_{11} - T_{45}$ |
| $X_6X_{12} - T_6$ | $X_5X_{10} - T_{16}$ | $X_{16}X_{23} - T_{26}$ | $X_9X_{15} - T_{36}$ | $X_9X_{20} - T_{46}$ |
| $X_9X_{19} - T_7$ | $X_1X_{10} - T_{17}$ | $X_3X_{13} - T_{27}$ | $X_2X_{13} - T_{37}$ | $X_4X_{21} - T_{47}$ |
| $X_6X_{10} - T_8$ | $X_{20}X_{23} - T_{18}$ | $X_{11}X_{20} - T_{28}$ | $X_{10}X_{23} - T_{38}$ | $X_7X_{23} - T_{48}$ |
| $X_8X_{18} - T_9$ | $X_2X_{23} - T_{19}$ | $X_{14}X_{22} - T_{29}$ | $X_3X_7 - T_{39}$ | $X_{14}X_{22} - T_{49}$ |
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### Description of the system

- 50 equations, 75 variables
- Polynomials $m_i - T_i$ with $m_i$ degree 2 monomial
- Goal: find all relations between the $m_i$
  $\iff$ find all polynomials $P(T_j)$ in the ideal

Tool: Gröbner bases

### Total degree grading

- difficult (~8h with Magma, intermediate basis in 4h)
- irregular behavior (highest deg. components not indep.)

### Weighted degree grading

- $\text{Weight}(T_i) = \text{Degree}(m_i) = 2$
- easier (~4h with Magma, intermediate basis in 0.1s)
- regular behavior

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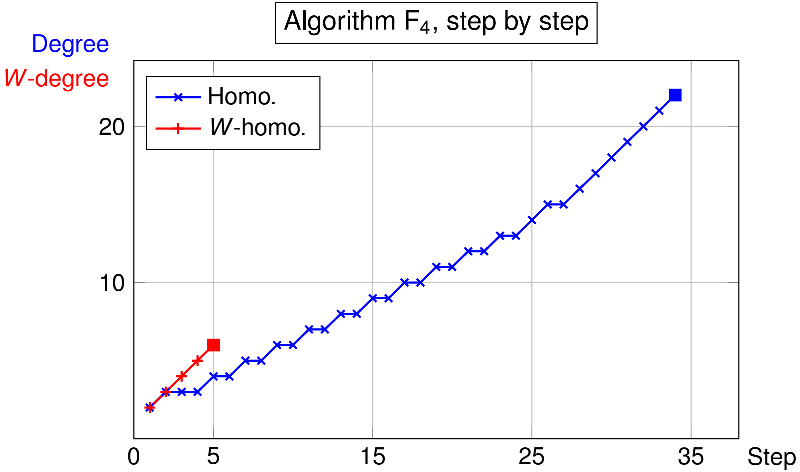
# A run of $F_4$ on the example

Ideal of relations between 50 monomials of degree 2 in 25 variables




- 50 equations of ($W$-)degree 2 in 75 variables
- GREVLEX ordering (e.g. for a 2-step strategy)
- Without weights: 3.9 h (34 steps reaching degree 22)
- With weights: 0.1 s (5 steps reaching $W$-degree 6)

# Gröbner bases and structured systems

**Polynomial system**

$$\begin{cases} f: X^2 + 2XY + Y^2 + X = 0 \\ g: X^2 - XY + Y^2 + Y - 1 = 0 \end{cases}$$

↓

**Gröbner basis**

$$\begin{cases} Y^3 + Y^2 - \frac{4}{9}X - \frac{2}{9}Y - \frac{4}{9} \\ X^2 + Y^2 + \frac{1}{3}X + \frac{2}{3}Y - \frac{2}{3} \\ XY + \frac{1}{3}X - \frac{1}{3}Y + \frac{1}{3} \end{cases}$$

### Problematic

Structured systems
→ Can we exploit it?

### Successfully studied structures

- Bihomogeneous (Dickenstein, Emiris, Faugère, Safey, Spaenlehauer...)
- Group symmetries (Colin, Faugère, Gatermann, Rahmany, Svartz...)
- Weighted homo. / Quasi-homo. ([Traverso 1996], [FSV 2013]. . . )

# Weighted homogeneous systems: definitions

## Definition (e.g. [Robbiano 1986], [Becker and Weispfenning 1993])

System of weights: $W = (w_1, \ldots, w_n) \in \mathbb{N}^n$

Weighted degree (or $W$-degree): $\deg_W(X_1^{\alpha_1} \ldots X_n^{\alpha_n}) = \sum_{i=1}^n w_i \alpha_i$

Weighted homogeneous polynomial: poly. with monomials of same $W$-degree

$\rightarrow$ Example: physical systems: Volume $=$ Area $\times$ Height

Weight 3 Weight 2 Weight 1

Given a general (not weighted homogeneous) system and a system of weights

Computational strategy: weighted-homogenize it as in the homogeneous case

Complexity estimates: consider the highest $W$-degree components of the system

- Enough to study weighted homogeneous systems

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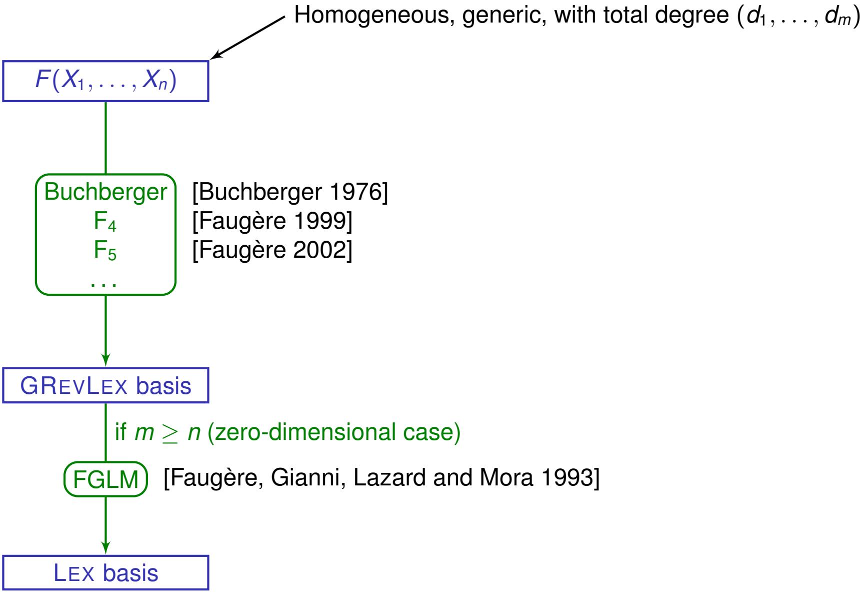
# Complexity for generic homogeneous systems



# Complexity for generic homogeneous systems

Homogeneous, generic, with total degree $(d_1, \ldots, d_m)$

$F(X_1, \ldots, X_n)$

$F_5$

- Highest degree $d_{\text{reg}} \leq \sum_{i=1}^{m} (d_i - 1) + 1$
- Size of the matrix at degree $d = \binom{n+d-1}{d}$

GREVLEX basis

if $m \geq n$ (zero-dimensional case)

FGLM

Number of solutions $= \prod_{i=1}^{n} d_i$ (Bézout bound)

LEX basis

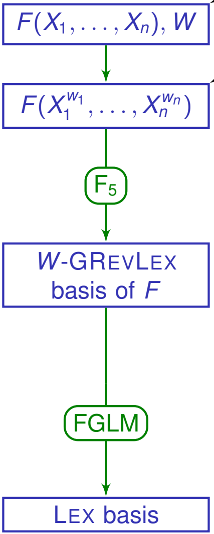
$$O\left(\binom{n+d_{\text{reg}}-1}{d_{\text{reg}}}^3 + n\left(\prod_{i=1}^{n} d_i\right)^3\right)$$

# Computational strategy for weighted homogeneous systems

$W = (w_1, \dots, w_n)$

$W$-Homogeneous, generic,
with $W$-degree $(d_1, \dots, d_m)$
(zero-dimensional: $m = n$)

Homogeneous, with total degree $(d_1, \dots, d_m)$



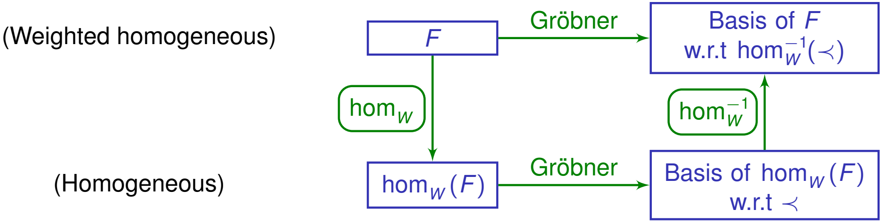
# Algorithms: from weighted homogeneous to homogeneous

## Transformation morphism

$$\begin{array}{rcll} \hom_W : & (\mathbb{K}[\mathbf{X}], W\text{-deg}) & \to & (\mathbb{K}[\mathbf{X}], \deg) \\ & f & \mapsto & f(X_1^{w_1}, \ldots, X_n^{w_n}) \end{array}$$

- Graded injective morphism
- Sends regular ("independent") sequences on regular sequences
- $\text{S-Pol}(\hom_W(f), \hom_W(g)) = \hom_W(\text{S-Pol}(f, g))$

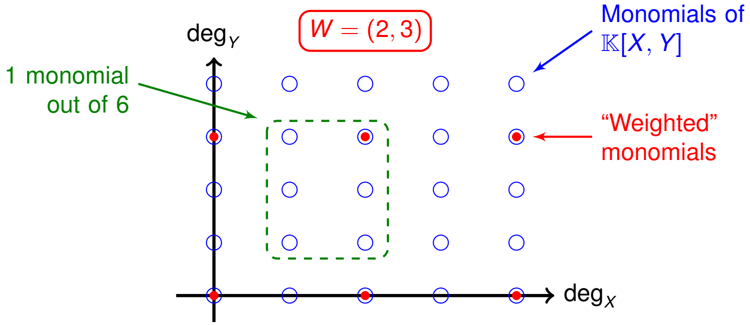
$\longrightarrow$ Good behavior w.r.t Gröbner bases



# Size of the Macaulay matrices

## Counting the monomials

- $\hom_W(F)$ lies in an algebra with a lot of useless monomials
- Count them: combinatorial object named Sylvester denumerants
- Result[1]: asymptotically $N_d \sim \frac{\#\text{Monomials of total degree } d}{\prod_{i=1}^{n} w_i}$




[1] Geir Agnarsson (2002). 'On the Sylvester denumerants for general restricted partitions'

# State of the art: complexity results [FSV 2013]

$W = (w_1, \dots, w_n)$

$F(X_1, \dots, X_n)$, $W$ ← $W$-Homogeneous, generic, with $W$-degree $(d_1, \dots, d_m)$ (zero-dimensional: $m = n$)

↓

$F(X_1^{w_1}, \dots, X_n^{w_n})$ ← Homogeneous, with total degree $(d_1, \dots, d_m)$

↓

$F_5$ ← 

- Highest $W$-degree

$$d_{W,\text{reg}} \leq \sum_{i=1}^{m} (d_i - w_i) + \max\{w_j\}$$

- Size of the matrix at $W$-degree $d \simeq \frac{1}{\prod_{i=1}^{n} w_i} \binom{n+d-1}{d}$

↓

$W$-GREVLEX basis of $F$

↓

FGLM ← Number of solutions $= \frac{\prod_{i=1}^{n} d_i}{\prod_{i=1}^{n} w_i}$ (weighted Bézout bound)

↓

LEX basis

$$O\left(\left(\frac{1}{\prod_{i=1}^{n} w_i}\right)^3 \left[\binom{n + d_{W,\text{reg}} - 1}{d_{W,\text{reg}}}^3 + n\left(\prod_{i=1}^{n} d_i\right)^3\right]\right)$$

# Main results: lifted hypotheses and sharper bound

$W = (w_1, \ldots, w_n)$

$F(X_1, \ldots, X_n)$, $W$ ← $W$-Homogeneous, generic, with $W$-degree $(d_1, \ldots, d_m)$ ($m < n$ or $m = n$ or $m > n$)

↓

$F(X_1^{w_1}, \ldots, X_n^{w_n})$ ← Homogeneous, with total degree $(d_1, \ldots, d_m)$

↓

F$_5$ ← Highest $W$-degree (if $m \leq n$)

$$d_{W,\mathrm{reg}} \leq \sum_{i=1}^{m} (d_i - w_i) + w_m$$

Size of the matrix at $W$-degree $d \simeq \frac{1}{\prod_{i=1}^{n} w_i} \binom{n+d-1}{d}$

↓

$W$-GRevLex basis of $F$

↓ (if $m \geq n$)

FGLM ← Number of solutions $= \frac{\prod_{i=1}^{n} d_i}{\prod_{i=1}^{n} w_i}$ (weighted Bézout bound)

$$O\left(\left(\frac{1}{\prod_{i=1}^{n} w_i}\right)^3 \left[\binom{n + d_{W,\mathrm{reg}} - 1}{d_{W,\mathrm{reg}}}^3 + n\left(\prod_{i=1}^{n} d_i\right)^3\right]\right)$$

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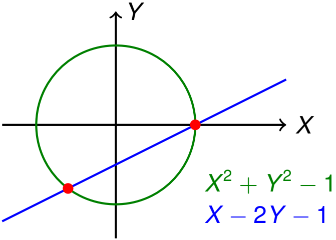
Lex basis

# Regular sequences

## Definition

$F = (f_1, \ldots, f_m)$ $W$-homo. $\in \mathbb{K}[\mathbf{X}]$ is regular iff

$$\begin{cases} \langle F \rangle \subsetneq \mathbb{K}[\mathbf{X}] \\ \forall i,\ f_i \text{ is no zero-divisor in } \mathbb{K}[\mathbf{X}]/\langle f_1, \ldots, f_{i-1} \rangle \end{cases}$$



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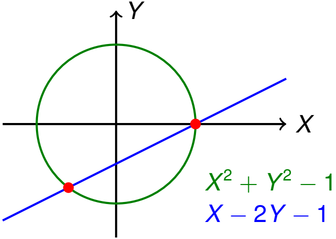



## Property [FSV 2013]

Regular sequences of $W$-homo. polynomials → Generic if $\neq \varnothing$

Regular sequences of $W$-homo. polynomials → Good properties → $F_5$-criterion, Hilbert series

# Properties of regular sequences

## Hilbert series

$$HS_{A/I}(T) = \sum_{d=0}^{\infty} (\text{rank defect of the } F_5 \text{ matrix at } W\text{-degree } d) \cdot T^d$$

## Properties

For regular sequences of $W$-homogeneous polynomials of $W$-degree $d_i$:

$$HS_{A/I}(T) = \frac{(1 - T^{d_1}) \cdots (1 - T^{d_m})}{(1 - T^{w_1}) \cdots (1 - T^{w_n})}$$

In dimension zero ($m = n$):

- Bézout bound on the degree: $D = \frac{\prod_{i=1}^{n} d_i}{\prod_{i=1}^{n} w_i}$
- Macaulay bound on $d_{\text{reg}}$ [FSV 2013] : $d_{\text{reg}} \leq \sum_{i=1}^{n} (d_i - w_i) + \max\{w_j\}$

# Can we do better? Yes, but not with the regularity alone.

## Positive dimension ($m < n$)

- Need to know what variables matter to the system
- Information not available from regularity

$\longrightarrow$ (Simultaneous) Noether position

## Dimension 0 ($m = n$)

- Macaulay's bound on $d_{\text{reg}}$ is not sharp
- $d_{\text{reg}}$ depends on the order of the variables:

| $W$ | $W$-degree | Macaulay's bound | $d_{\text{reg}}$ | $F_4$ DRL time |
|---|---|---|---|---|
| $(20, 5, 5, 1)$ | $(60, 60, 60, 60)$ | 229 | 210 | 471s |
| $(1, 5, 5, 20)$ | $(60, 60, 60, 60)$ | 229 | 220 | 916s |

$\longrightarrow$ Simultaneous Noether position

## Overdetermined systems ($m > n$)

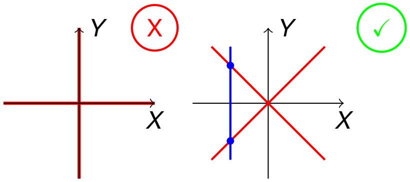
- No regular sequence $\longrightarrow$ Semi-regularity

# Noether position

## Noether position

$F = (f_1, \ldots, f_m) \in \mathbb{K}[X_1, \ldots, X_n]$, $m \le n$

- Noether position:
  $(F, X_{m+1}, \ldots, X_n)$ regular
- simultaneous Noether position:
  $(f_1, \ldots, f_j)$ in NP for all $j$'s




## Properties

- Generic if not empty
- Valid under generic change of coordinates for "nice" systems of weights
- Relevant property for fine-grained complexity (structure lemma [Bardet 2004])
- For a $W$-homogeneous sequence in simultaneous Noether position:

$$d_{\text{reg}} \le \sum_{i=1}^{m} (d_i - w_i) + w_m \qquad (\text{sharp if } w_m = 1)$$

# Semi-regular sequences

## Semi-regular sequences

- If $m > n$, reductions to zero cannot be eliminated.
- Semi-regular sequence: all reductions to zero are at high degrees
- Hilbert series of a semi-regular homogeneous sequence:

$$\mathrm{HS}_{A/I}(T) = \left\lfloor \frac{(1-T^{d_1})\cdots(1-T^{d_m})}{(1-T)^n} \right\rfloor \text{ (series truncated to the first coefficient } \leq 0)$$

- For $W$-homogeneous systems, only true for “nice” systems of weights
- Main consequence: asymptotic estimate of the degree of regularity [Bardet 2004]

## Fröberg’s conjecture

Semi-regular sequences are generic.

Proved for:

- $n = 2$
- $n = 3$ for large fields
- $m = n + 1$ in characteristic 0

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- Semi-regular sequence: all reductions to zero are at high degrees
- Hilbert series of a semi-regular $W$-homogeneous sequence:

$$\mathrm{HS}_{A/I}(T) = \left\lfloor \frac{(1-T^{d_1})\cdots(1-T^{d_m})}{(1-T^{w_1})\cdots(1-T^{w_n})} \right\rfloor \text{ (series truncated to the first coefficient } \leq 0)$$

- For $W$-homogeneous systems, only true for "nice" systems of weights
- Main consequence: asymptotic estimate of the degree of regularity [Bardet 2004]

## Fröberg's conjecture

Semi-regular sequences are generic.

Proved for:

- $n = 2$
- $n = 3$ for large fields
- $m = n + 1$ in characteristic 0

# Complexity

## Input

- $W = (w_1, \ldots, w_n)$
- $F = (f_1, \ldots, f_m) \in \mathbb{K}[X_1, \ldots, X_n]$ generic $W$-homogeneous

## Complexity of $\mathrm{F}_5$

$$\left(\frac{1}{\prod_{i=1}^n w_i}\right)^3 \binom{n + d_{\mathrm{reg}} - 1}{d_{\mathrm{reg}}}^3$$

- Asymptotic gain from the size of the matrices
- Practical gain from the weighted Macaulay bound ($d_{\mathrm{reg}}$)

## Complexity of FGLM ($m = n$)

$$\left(\frac{1}{\prod_{i=1}^n w_i}\right)^3 n \left(\prod_{i=1}^n d_i\right)^3$$

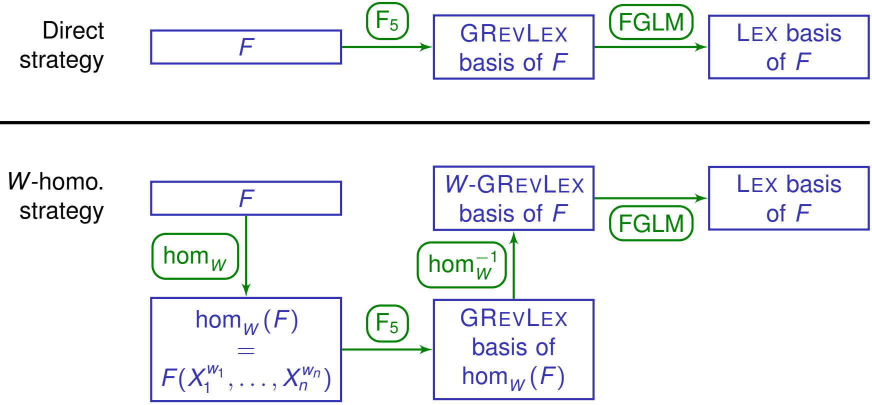
- Asymptotic gain from the weighted Bézout bound (number of solutions)

# Benchmarking

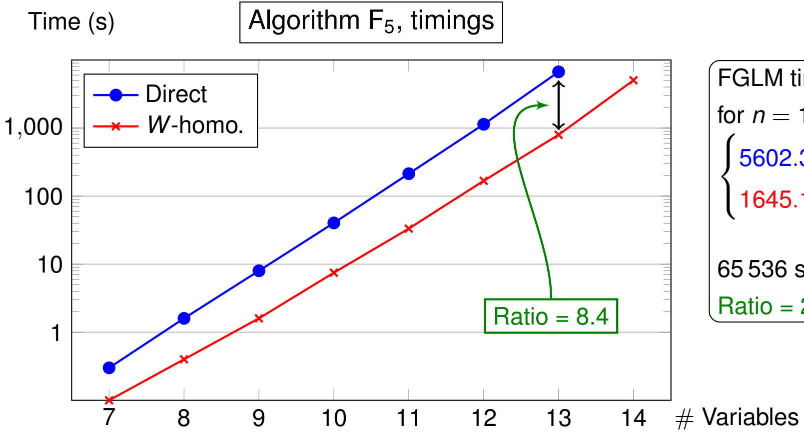
$F$ : 0-dim. affine system with a weighted homogeneous structure

$$f_i = \sum_{\alpha} c_\alpha m_\alpha \text{ with } \deg_W(m_\alpha) \leq d_i$$

Assumption: the highest $W$-degree components are regular (e.g. if $F$ is generic)



# Benchmarks for generic systems




FGLM timing for $n = 13$:

$\begin{cases} 5602.3\,\text{s} \\ 1645.1\,\text{s} \end{cases}$

65 536 solutions

Ratio = 2.1

- Generic systems in $n$ variables with $\begin{cases} \text{weights } W = (2, \ldots, 2, 1, 1) \\ W\text{-degree } D = (4, \ldots, 4) \end{cases}$
- Number of solutions: $2^{n+2}$
- Benchmarks obtained with FGb : $\begin{cases} F_5 \text{ [Faugère 2002]} \\ \text{SPARSEFGLM [Faugère and Mou 2013]} \end{cases}$

# The story is not over...

## Sometimes, “normally” faster...

- Generic complete intersection (GREVLEX): 13 min. vs. 1h45 (speed-up: 8)
- Relations between monomials (elim.): 4h vs 8h (speed-up: 2)
- Relations between 14 invariants of the cyclic-5 group (elim.): 40 min. vs 10h (speed-up: 16)

## ... sometimes, faster than that...

- Relations between monomials (GREVLEX): 0.1s vs 4h (speed-up: 144 000)

## ... and sometimes, same speed.

- Relations between monomials (elim. from GREVLEX)
- Elimination on generic systems (elim.)

# Conclusion and perspectives

## What we have done

- Theoretical results for $W$-homogeneous systems under generic assumptions
- Complexity results for $F_5$ for positive-dim. systems and overdetermined systems
  - Bound on the maximal degree reached by the $F_5$ algorithm
  - Complexity overall divided by $(\prod w_i)^3$

## Consequences

Wide range of potential applications:

- Polynomial inversion, implicitization (positive dimension)
- Cryptography (overdetermined)

## Perspectives

- Timings still not completely understood
- Affine systems: find the most appropriate system of weights
- Additional structure: $W$-homo. for several systems of weights, weights $\leq 0$...

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Thank you for your attention!