On The Complexity Of Computing Gröbner Bases For Quasi-Homogeneous Systems

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Motivation

Discrete Logarithm Problem (Faugère, Gaudry, Huot, Renault 2013)

$$0 = \begin{bmatrix} 7871 \\ 18574 \\ 14294 \\ 32775 \\ 20289 \end{bmatrix} e_5^{16} + \begin{bmatrix} 53362 \\ 50900 \\ 36407 \\ 58813 \\ 20802 \end{bmatrix} \tilde{e}_1^8 + \begin{bmatrix} 26257 \\ 128 \\ 3037 \\ 38424 \\ 41456 \end{bmatrix} \tilde{e}_1^7 \tilde{e}_2 + \begin{bmatrix} 25203 \\ 23117 \\ 28918 \\ 29298 \\ 56353 \end{bmatrix} \tilde{e}_1^6 \tilde{e}_2^2 + \begin{bmatrix} 19817 \\ 29737 \\ 52187 \\ 36574 \\ 46683 \end{bmatrix} \tilde{e}_1^5 \tilde{e}_2^3 + \begin{bmatrix} 9843 \\ 3752 \\ 27006 \\ 64195 \\ 63059 \end{bmatrix} \tilde{e}_1^4 \tilde{e}_2^4 + \begin{bmatrix} 11204 \\ 25459 \\ 58263 \\ 17964 \\ 57146 \end{bmatrix} \tilde{e}_1^3 \tilde{e}_2^5$$

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Description of the system

- Ideal invariant under the group $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes \mathfrak{S}_n$,
  rewritten with the invariants:
  $$\begin{cases} \tilde{e}_i := e_i(x_1^2, \ldots, x_n^2) \ \ (1 \leq i \leq n-1) \\ e_n(x_1, \ldots, x_n) \end{cases}$$
- n equations of degree 2^{n-1} in $\mathbb{F}_q[\tilde{e}_1, \ldots, \tilde{e}_{n-1}, e_n]$
- 1 DLP $=$ thousands of such systems

Goal: solve the system
$\iff$ compute a Gröbner basis

- Total degree grading
  $\rightarrow$ difficult (intractable with Magma)
  $\rightarrow$ non regular
- Weighted degree grading
  $\text{Weight}(\tilde{e}_i) = 2 \cdot \text{Weight}(e_i)$
  $\rightarrow$ easier
  $\rightarrow$ regular
- Two questions:
  - Algorithms for this structure?
  - Complexity estimates?

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Gröbner bases and structured systems

Polynomial system

$$\begin{cases} f: X^2 + 2XY + Y^2 + X = 0 \\ g: X^2 - XY + Y^2 + Y - 1 = 0 \end{cases}$$

↓

Gröbner basis

$$\begin{cases} Y^3 + Y^2 - \frac{4}{9}X - \frac{2}{9}Y - \frac{4}{9} \\ X^2 + Y^2 + \frac{1}{3}X + \frac{2}{3}Y - \frac{2}{3} \\ XY + \frac{1}{3}X - \frac{1}{3}Y + \frac{1}{3} \end{cases}$$

Problematic

Structured systems
$\rightarrow$ Can we exploit it?

Successfully studied structures

- Bihomogeneous (Dickenstein, Emiris, Faugère, Safey, Spaenlehauer...)
- Group symmetries (Colin, Faugère, Gatermann, Rahmany, Svartz...)
- Quasi-homogeneous? ([Traverso 1996]. . .)

Quasi-homogeneous systems: définitions

Definition (e.g. [Robbiano 1986], [Becker and Weispfenning 1993])

System of weights: $W = (w_1, \ldots, w_n) \in \mathbb{N}^n$

Weighted degree (or W-degree): $\deg_W(X_1^{\alpha_1} \ldots X_n^{\alpha_n}) = \sum_{i=1}^{n} w_i \alpha_i$

Quasi-homogeneous polynomial: poly. containing only monomials of same W-degree

$\rightarrow$ Example: physical systems: Volume = Area × Height

Weight 3 Weight 2 Weight 1

Given a general (non-quasi-homogeneous) system and a system of weights

Computational strategy: quasi-homogenize it as in the homogeneous case

Complexity estimates: consider the highest-W-degree components of the system

- Enough to study quasi-homogeneous systems

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Weight 3 (Volume), Weight 2 (Area), Weight 1 (Height)

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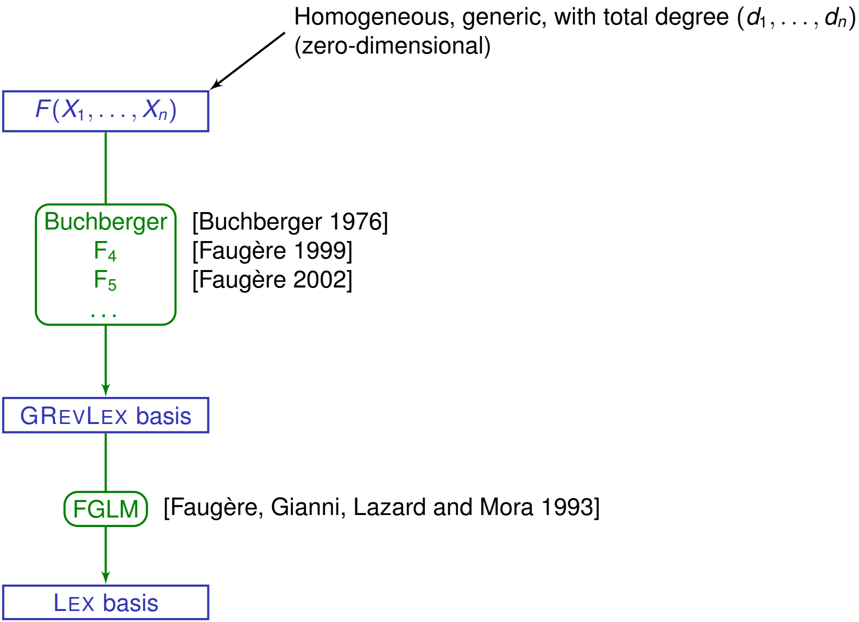
Complexity for generic homogeneous systems



Complexity for generic homogeneous systems

Homogeneous, generic, with total degree $(d_1, \ldots, d_n)$ (zero-dimensional)

$F(X_1, \ldots, X_n)$

F_5 ←
- Highest degree $d_{\text{reg}} \leq \sum_{i=1}^{n} (d_i - 1) + 1$
- Size of the matrix at degree $d = \binom{n+d-1}{d}$

GRevLex basis

FGLM ← Number of solutions $= \prod_{i=1}^{n} d_i$ (Bézout bound)

Lex basis

$$O\left(\binom{n+d_{\text{reg}}-1}{d_{\text{reg}}}^3 + n\left(\prod_{i=1}^{n} d_i\right)^3\right)$$

Main results: strategy and complexity results

$W = (w_1, \ldots, w_n)$

$F(X_1, \ldots, X_n)$, W ← W-Homogeneous, generic, with W-degree $(d_1, \ldots, d_n)$ (zero-dimensional)

↓

$F(X_1^{w_1}, \ldots, X_n^{w_n})$ ← Homogeneous, with total degree $(d_1, \ldots, d_n)$

↓

F_5 ← Highest W-degree

$$d_{W,\mathrm{reg}} \leq \sum_{i=1}^{n}(d_i - 1) + 1 - \sum_{i=1}^{n}(w_i - 1) + w_n - 1$$

Size of the matrix at W-degree $d \simeq \dfrac{1}{\prod_{i=1}^{n} w_i}\dbinom{n+d-1}{d}$

↓

W-GRevLex basis of F

↓

FGLM ← Number of solutions $= \dfrac{\prod_{i=1}^{n} d_i}{\prod_{i=1}^{n} w_i}$ (weighted Bézout bound)

$$O\left(\left(\frac{1}{\prod_{i=1}^{n} w_i}\right)^3 \left[\binom{n+d_{W,\mathrm{reg}}-1}{d_{W,\mathrm{reg}}}^3 + n\left(\prod_{i=1}^{n} d_i\right)^3\right]\right)$$

↓

Lex basis

Roadmap

Input

- $W = (w_1, \ldots, w_n)$ system of weights
- $F = (f_1, \ldots, f_m)$ generic sequence of W-homogeneous polynomials with W-degree $(d_1, \ldots, d_m)$

General roadmap:

1. Find a generic property with good complexity estimates
   - Regular sequences (dimension 0, $m = n$)
   - Noether position (positive dimension, $m \leq n$)
   - . . . Semi-regular sequences (dimension 0, $m > n$)
2. Design new algorithms to take advantage of this structure
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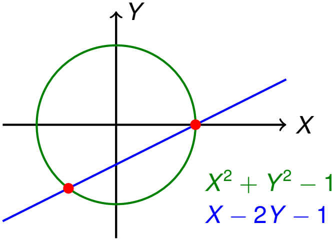
Regular sequences

Definition

$F = (f_1, \dots, f_m)$ homo. $\in \mathbb{K}[\mathbf{X}]$ is regular iff

$$\begin{cases} \langle F \rangle \subsetneq \mathbb{K}[\mathbf{X}] \\ \forall i,\ f_i \text{ is no zero-divisor in } \mathbb{K}[\mathbf{X}]/\langle f_1, \dots, f_{i-1} \rangle \end{cases}$$



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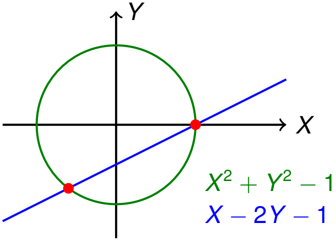

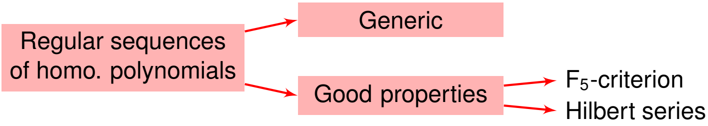

Regular sequences

Definition

$F = (f_1, \ldots, f_m)$ quasi-homo. $\in \mathbb{K}[\mathbf{X}]$ is regular iff

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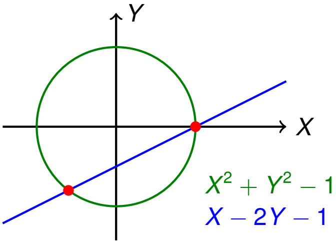
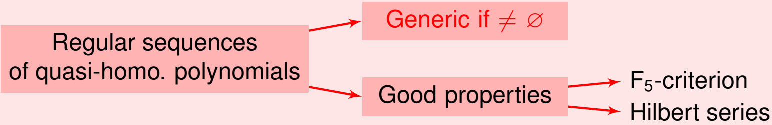


Result (Faugère, Safey, V.)



Properties of regular sequences

Hilbert series

$$HS_{A/I}(T) = \sum_{d=0}^{\infty} (\text{rank defect of the } F_5 \text{ matrix at degree } d) \cdot T^d$$

Properties

For regular sequences of homogeneous polynomials of degree d_i:

$$HS_{A/I}(T) = \frac{(1 - T^{d_1}) \cdots (1 - T^{d_m})}{(1 - T)^n}$$

In zero dimension ($m = n$):

- Bézout bound on the degree: $D = \prod_{i=1}^{n} d_i$
- Macaulay bound on the degree of regularity: $d_{\text{reg}} \leq \sum_{i=1}^{n} (d_i - 1) + 1$

Properties of regular sequences

Hilbert series

$$\mathrm{HS}_{A/I}(T) = \sum_{d=0}^{\infty} (\text{rank defect of the } F_5 \text{ matrix at } W\text{-degree } d) \cdot T^d$$

Properties

For regular sequences of W-homogeneous polynomials of W-degree d_i:

$$\mathrm{HS}_{A/I}(T) = \frac{(1 - T^{d_1}) \cdots (1 - T^{d_m})}{(1 - T^{w_1}) \cdots (1 - T^{w_n})}$$

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Limitations

Limitations of the regularity

- $m < n$ (positive dimension): no real information
- $m = n$ (zero dimension, complete intersection)
  - exact formula for d_{reg}?
  - d_{reg} depends on the order of the variables
  - Hilbert series: independent from that order
- $m > n$ (cryptography): no regular sequence

$\Longrightarrow$ Additional properties

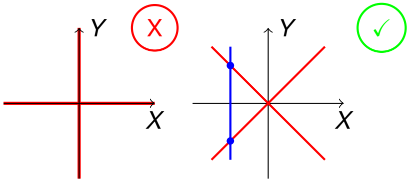
- $m < n$: Noether position
- $m = n$: simultaneous Noether position
- $m > n$: semi-regular sequences

Noether position

Noether position

$F = (f_1, \ldots, f_m) \in \mathbb{K}[X_1, \ldots, X_n]$, $m \leq n$

- Noether position:
  $(F, X_{m+1}, \ldots, X_n)$ regular
- simultaneous Noether position:
  $(f_1, \ldots, f_j)$ in NP for all j's



Properties

- Generic if not empty
- Valid under generic change of coordinates for "nice" systems of weights
- Relevant property for fine-grained complexity (structure lemma [Bardet 2004])
- For a zero-dim. W-homogeneous sequence in simultaneous Noether position:

$$d_{\text{reg}} = \sum_{i=1}^{n} (d_i - w_i) + w_n$$

Semi-regular sequences

Semi-regular sequences

- If $m > n$, reductions to zero cannot be eliminated.
- Semi-regular sequence: all reductions to zero are at high degrees
- Hilbert series of a semi-regular homogeneous sequence:

$$\mathrm{HS}_{A/I}(T) = \left\lfloor \frac{(1-T^{d_1})\cdots(1-T^{d_m})}{(1-T)^n} \right\rfloor \text{ (series truncated to the first coefficient } \leq 0\text{)}$$

- For W-homogeneous systems, only true for “nice” systems of weights
- Main consequence: asymptotic estimate of the degree of regularity [Bardet 2004]

Fröberg’s conjecture

Semi-regular sequences are generic.

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Fröberg's conjecture

Semi-regular sequences are generic.

Roadmap

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- $W = (w_1, \ldots, w_n)$ system of weights
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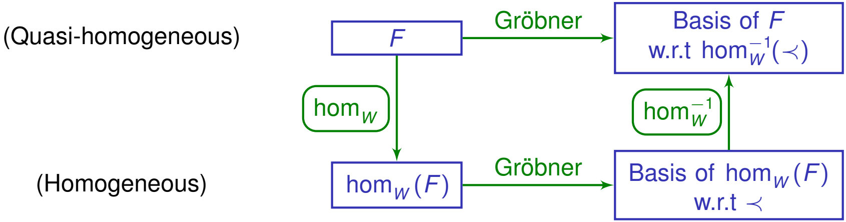
Algorithms: from quasi-homogeneous to homogeneous

Transformation morphism

$$\begin{array}{rccc} \hom_W : & (\mathbb{K}[\mathbf{X}], W\text{-deg}) & \to & (\mathbb{K}[\mathbf{X}], \deg) \\ & f & \mapsto & f(X_1^{w_1}, \dots, X_n^{w_n}) \end{array}$$

- Graded injective morphism
- Sends regular sequences on regular sequences
- $\text{S-Pol}(\hom_W(f), \hom_W(g)) = \hom_W(\text{S-Pol}(f, g))$

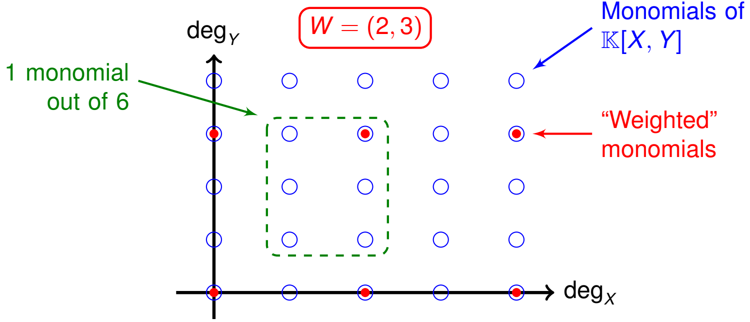
$\longrightarrow$ Good behavior w.r.t Gröbner bases



Size of the Macaulay matrices

Counting the monomials

- $\hom_W(F)$ lies in an algebra with a lot of useless monomials
- Count them: combinatorial object named Sylvester denumerants
- Result[1]: asymptotically $N_d \sim \dfrac{\#\text{Monomials of total degree } d}{\prod_{i=1}^{n} w_i}$

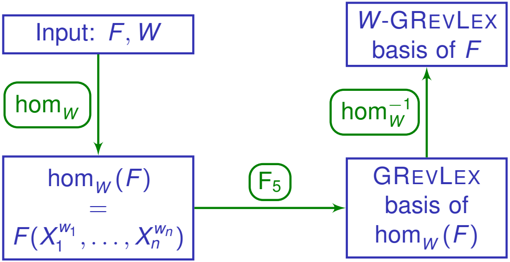


[1] Geir Agnarsson (2002). 'On the Sylvester denumerants for general restricted partitions'

Adapting the algorithms

Detailed strategy

- F_5 algorithm on the homogenized system
- FGLM algorithm on the quasi-homogeneous system



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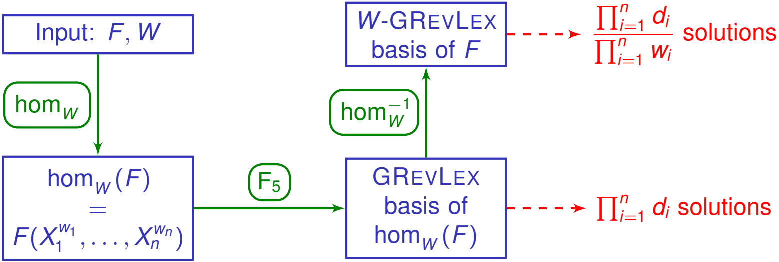

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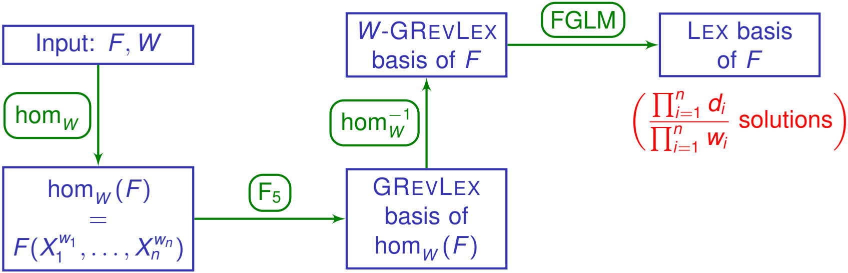

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   - Regular sequences (dimension 0, $m = n$)
   - Noether position (positive dimension, $m \leq n$)
   - ...Semi-regular sequences (dimension 0, $m > n$)
2. Design new algorithms to take advantage of this structure
   - Adapt algorithms for the homogeneous case to the quasi-homogeneous case
3. Conclusion: complexity results

Complexity

Input

- $W = (w_1, \dots, w_n)$
- $F = (f_1, \dots, f_n) \in \mathbb{K}[X_1, \dots, X_n]$ generic W-homogeneous

Complexity of F_5

$$\left(\frac{1}{\prod_{i=1}^n w_i}\right)^3 \binom{n + d_{\mathrm{reg}} - 1}{d_{\mathrm{reg}}}^3$$

- Asymptotic gain from the size of the matrices
- Practical gain from the weighted Macaulay bound (d_{reg})

Complexity of FGLM

$$\left(\frac{1}{\prod_{i=1}^n w_i}\right)^3 n \left(\prod_{i=1}^n d_i\right)^3$$

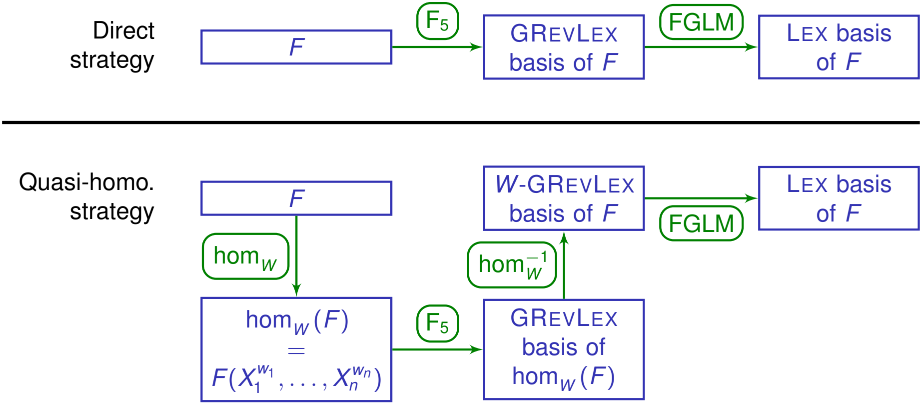
- Asymptotic gain from the weighted Bézout bound (number of solutions)

Benchmarking

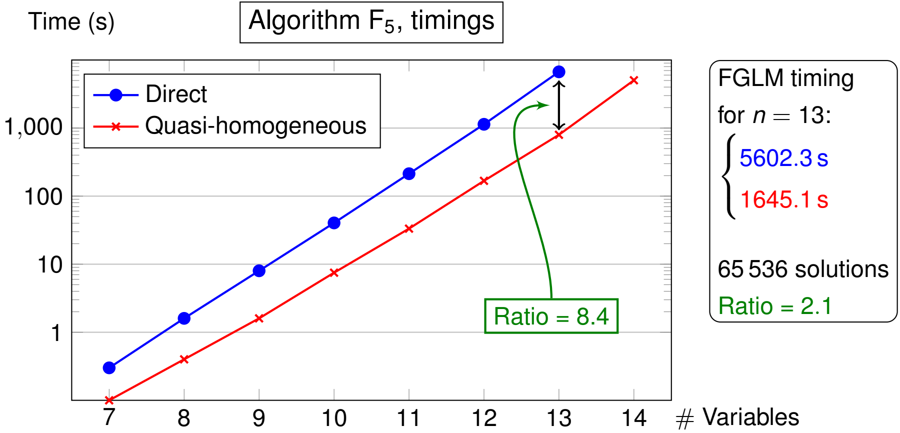
F : affine system with a quasi-homogeneous structure

$$f_i = \sum_{\alpha} c_\alpha m_\alpha \text{ with } \deg_W(m_\alpha) \leq d_i$$

Assumption: the highest W-degree components are regular (e.g. if F is generic)

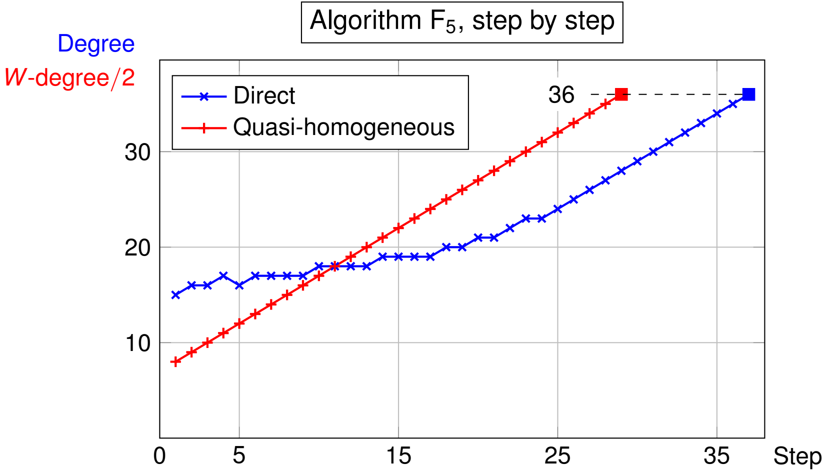


Benchmarks for generic systems



- Generic systems in n variables with $\begin{cases} \text{weights } W = (2, \ldots, 2, 1, 1) \\ W\text{-degree } D = (4, \ldots, 4) \end{cases}$
- Number of solutions: 2^{n+2}
- Benchmarks obtained with FGb : $\begin{cases} F_5 \text{ [Faugère 2002]} \\ \text{SPARSEFGLM [Faugère and Mou 2013]} \end{cases}$

A run of F_5 on the DLP example

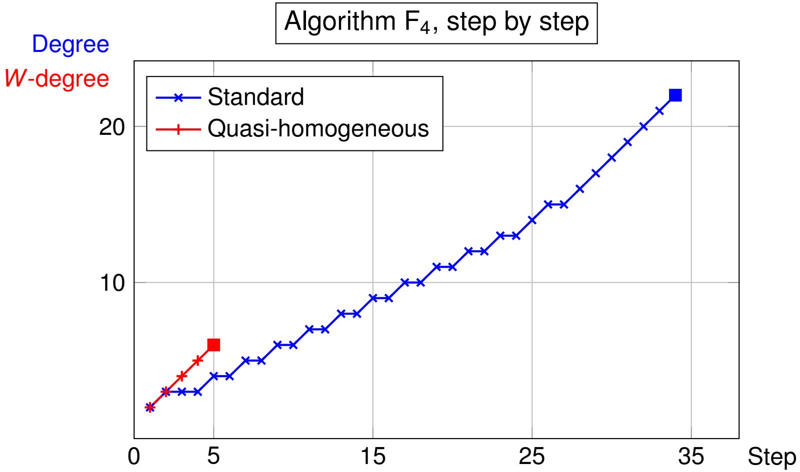


- 5 equations of W-degree $(16, \ldots, 16)$ in 5 variables with $W = (2, \ldots, 2, 1)$
- 65 536 solutions
- Timings:

Magma (F_4)	> 12 h	1.7 h	Speed-up: 9.3
FGb (F_5)	12 297 s	567 s	Speed-up: 21.7

A run of F_4 on an inversion example

Ideal of relations between 50 monomials of degree 2 in 25 variables



- 50 equations of (W-)degree 2 in 75 variables
- GRevLex ordering (e.g. for a 2-step strategy)
- Without weights: 3.9 h (34 steps reaching degree 22)
- With weights: 0.1 s (5 steps reaching W-degree 6)

Conclusion

What we have done

- Theoretical results for quasi-homogeneous systems under generic assumptions
- Computational strategy for quasi-homogeneous systems
- Complexity results for F_5 and FGLM for this strategy
  - Bound on the maximal degree reached by the F_5 algorithm
  - Complexity overall divided by $(\prod w_i)^3$

Consequences

- Successfully applied to a cryptographical problem
- Wide range of potential applications

Perspectives

- Affine systems: find the most appropriate system of weights (e.g for the DLP, how to choose the weights of the e_i's?)
- Additional structure: quasi-homo. for several systems of weights, weights ≤ 0...

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Thank you for your attention!