Computing Gröbner bases for quasi-homogeneous systems

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Motivations

Polynomial system

$f_1(\mathbf{X}) = \cdots = f_m(\mathbf{X}) = 0$

→ Gröbner basis →

Parametrization of the solutions

$$\mathbf{q}(\mathbf{T}) = 0$$
$$\mathbf{X} = \mathbf{p}(\mathbf{T})$$

Applications:

- Cryptography
- Physics, industry...
- Theory (algo. geometry)

Difficult problem

- NP-hard in finite field
- Exponential number of solutions

Problem:
Exploit the structures of the system

Examples of successfully studied structures:

- Homogeneous
- Bihomogeneous: [**FSS10b**]
- Group symmetries: e.g [**FS12**]
- Quasi-homogeneous

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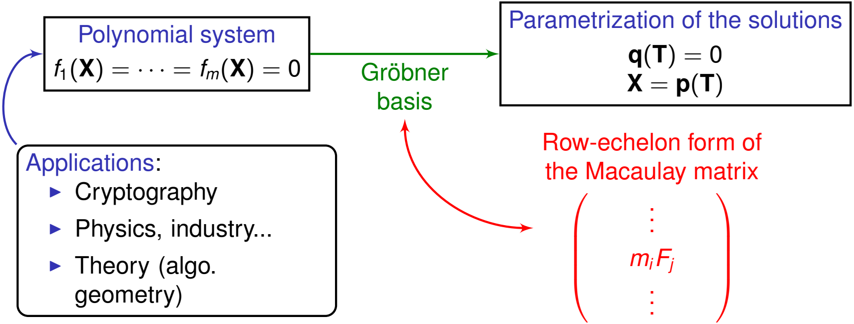

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Definitions of quasi-homogeneous systems

Definition

System of weights: $W = (w_1, \ldots, w_n) \in \mathbb{N}^n$

Weighted degree: $\deg_W(X_1^{\alpha_1} \ldots X_n^{\alpha_n}) = \sum_{i=1}^{n} w_i \alpha_i$

Quasi-homogeneous polynomial: poly. containing only monomials of same W-degree

e.g. $X^2 + XY^2 + Y^4$ for $W = (2, 1)$

- Homogeneous systems are W-homogeneous with weights $(1, \ldots, 1)$.

Applications

Physical system

Volume $=$ Area $\times$ Height

Weight 3 (Volume), Weight 2 (Area), Weight 1 (Height)

Polynomial inversion

$$\begin{cases} X = T^2 + U^2 & \text{Weight 2} \\ Y = T^3 - TU^2 & \text{Weight 3} \\ Z = T + 2U & \text{Weight 1} \end{cases} \longrightarrow P(X, Y, Z) = 0$$

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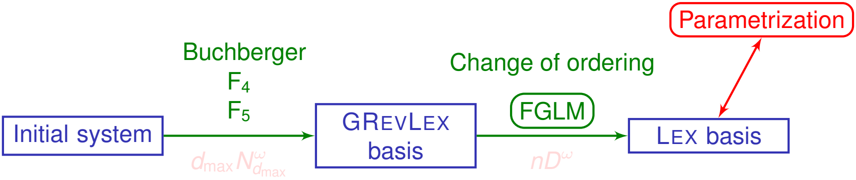
Usual two-steps strategy in the zero-dimensional case



Relevant complexity parameters

- $d_{\max}$ = highest degree reached by F_5
  Less than the degree of regularity d_{reg}.
  For generic homo. systems:
  $$d_{\text{reg}} = \sum_{i=1}^{n}(d_i - 1) + 1 \text{ [**Lazard83**]}$$
- D = degree of the ideal
  $$= \text{number of solutions in dim. } 0$$
  $$= \prod_{i=1}^{n} d_i \text{ (homo. generic case)}$$
- N_d (Macaulay matrix at degree d) → N_d

For an homogeneous system:

$$N_d = \binom{n+d-1}{d}$$

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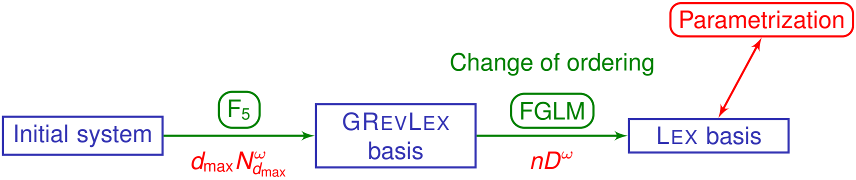

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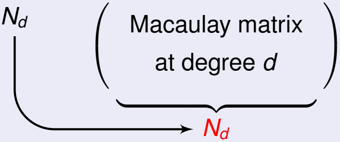

For an homogeneous system:

$$N_d = \binom{n+d-1}{d}$$

Main results

Adaptation of the usual strategy, so that we still have the complexity:

- $C_{F_5} = O\left(d_{\text{reg}} N_{d_{\text{reg}}}^{\omega}\right)$
- $C_{\text{FGLM}} = O\left(nD^{\omega}\right)$

with estimations of the parameters for generic quasi-homogeneous systems:

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- $d_{\text{reg}} = \displaystyle\sum_{i=1}^{n}(d_i - w_i) + \max\{w_j\}$
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Remark

If we set the weights to $(1, \ldots, 1)$,
we recover the usual values for homogeneous systems.

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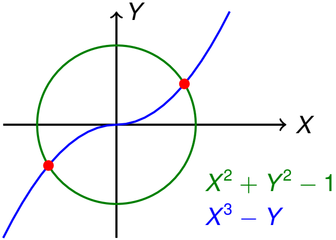
Regular sequences

Definition

$F = (f_1, \ldots, f_m)$ quasi-homo. $\in \mathbb{K}[\mathbf{X}]$ is regular iff

$$\begin{cases} \langle F \rangle \subsetneq \mathbb{K}[\mathbf{X}] \\ \forall i,\ f_i \text{ is no zero-divisor in } \mathbb{K}[\mathbf{X}]/\langle f_1, \ldots, f_{i-1} \rangle \end{cases}$$

For affine systems: defined w.r.t the highest weighted-degree components.



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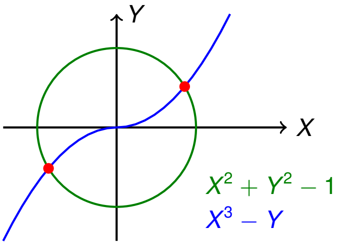

Generic sequences of homo. polynomials → Generic

Generic sequences of homo. polynomials → Good properties → F_5-criterion; Complexity results

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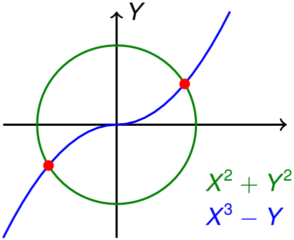

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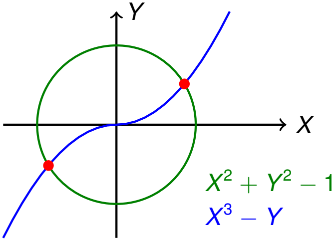

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Why this condition?

- $W = (2, 3),\ d_1 = 4,\ d_2 = 4$: no regular sequence
- $W = (2, 3),\ d_1 = 6,\ d_2 = 6$: (X_1^3, X_2^2) is regular, regular sequences are generic

Hilbert series, degree and degree of regularity

Hilbert series of an ideal

The Hilbert series of a (quasi-)homogeneous ideal is defined as the generating series of the rank defects in the Macaulay matrices of successive degrees.

$$\mathrm{HS}_I(t) = \sum_{d=0}^{\infty} \dim_{K-ev} \left(K[\mathbf{X}]/I\right)_d t^d$$

Expression for a zero-dimensional regular sequence:

$$\mathrm{HS}_I(t) = \frac{(1-t^{d_1})\cdots(1-t^{d_n})}{(1-t)\cdots(1-t)} = (1+\cdots+t^{d_1-1})\cdots(1+\cdots+t^{d_n-1})$$

Bézout and Macaulay bounds

- Bézout bound: $D = \mathrm{HS}_I(t := 1) = \prod_{i=1}^{n} d_i$
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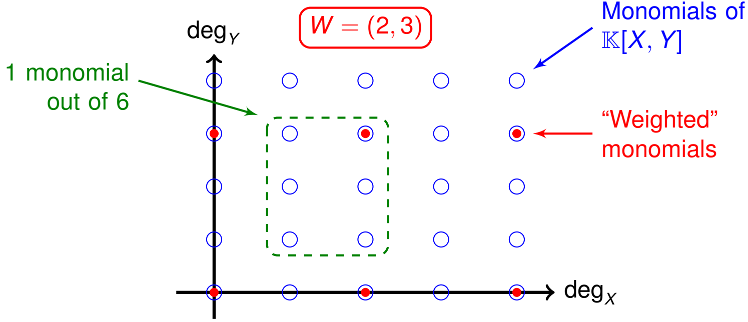
Size of the Macaulay matrices

- Need to count the monomials with a given W-degree
- Combinatorial object named Sylvester denumerants
- Result[1]: asymptotically $N_d \sim \frac{\#\text{Monomials of total degree } d}{\prod_{i=1}^{n} w_i}$



[1] **Geir02**.

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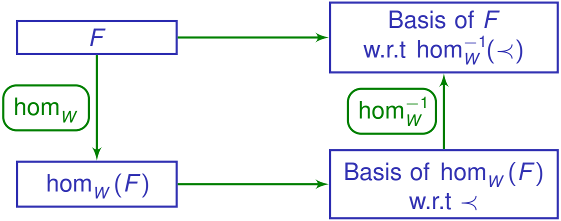
Homogenization morphism

$$\begin{array}{rccc} \hom_W : & (\mathbb{K}[\mathbf{X}], W\text{-deg}) & \to & (\mathbb{K}[\mathbf{X}], \deg) \\ & f & \mapsto & f(X_1^{w_1}, \ldots, X_n^{w_n}) \end{array}$$

- Graded injective morphism.
- Sends regular sequences onto regular sequences
- Good behavior w.r.t Gröbner bases



The W-GRevLex ordering

Analogous to the GRevLex ordering, except monomials are selected according to their W-degree instead of total degree.

Detailed strategy

Adapting the algorithms

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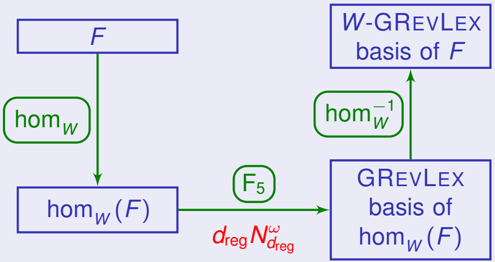

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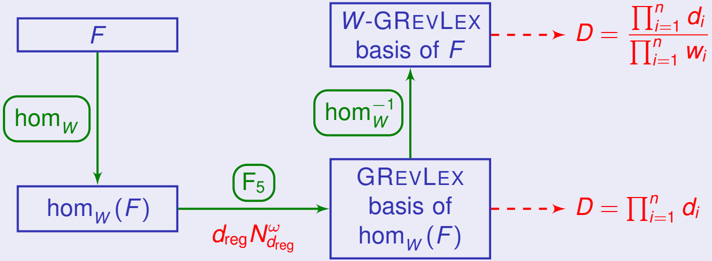

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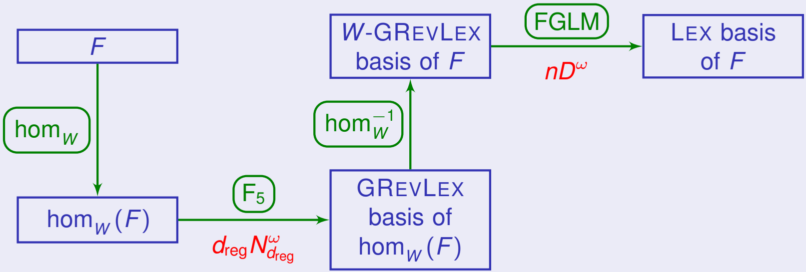

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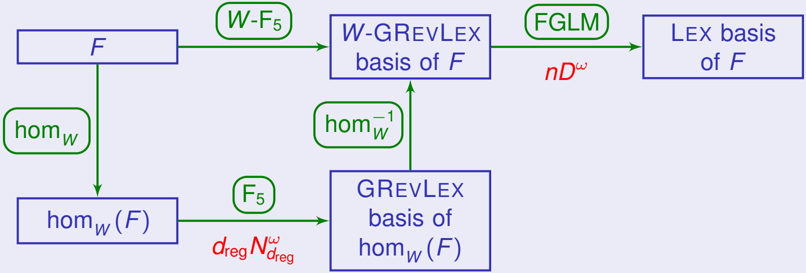

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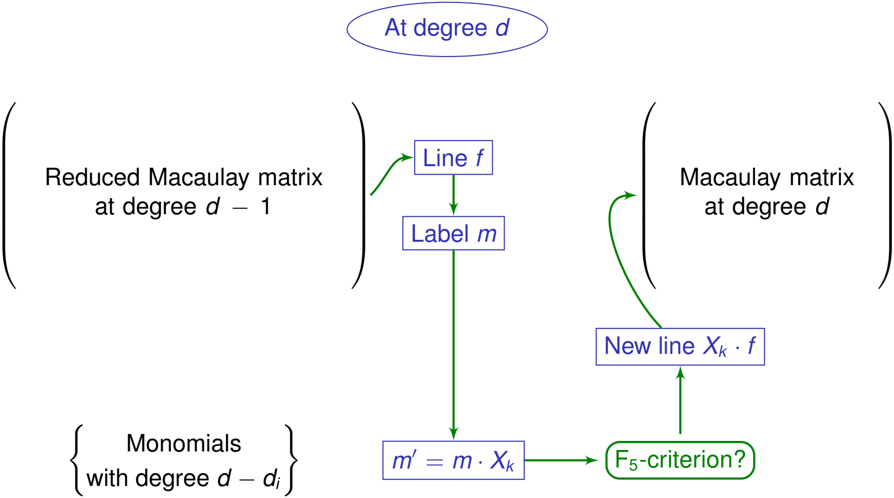

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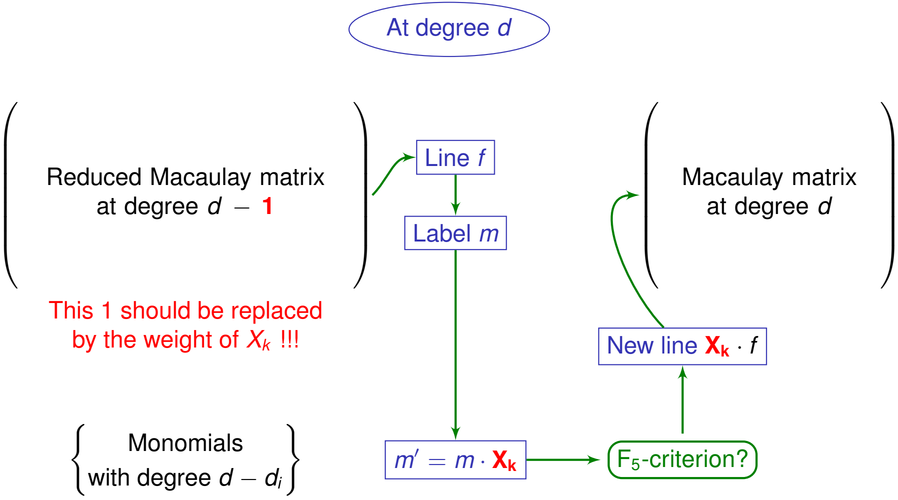

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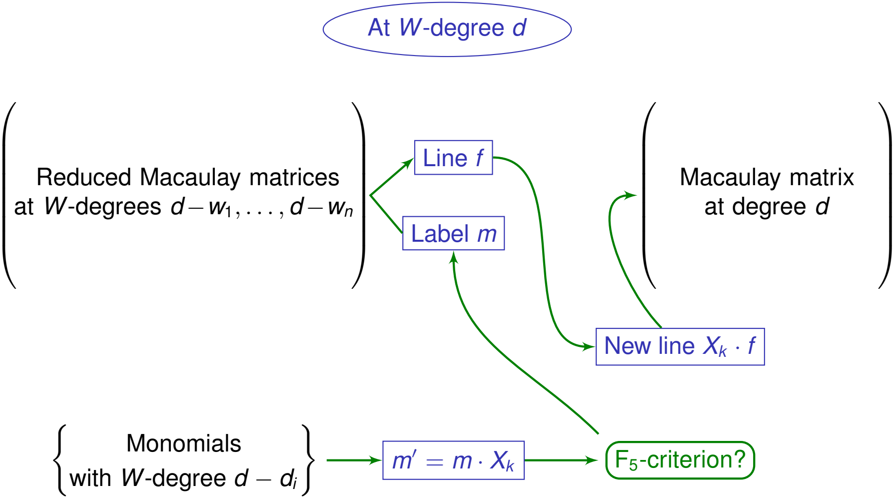

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Main results

Adaptation of the usual strategy, so that we still have the complexity:

- $C_{F_5} = O\left(d_{\text{reg}} N_{d_{\text{reg}}}^{\omega}\right)$
- $C_{\text{FGLM}} = O\left(nD^{\omega}\right)$

with estimations of the parameters for generic quasi-homogeneous systems:

- $D = \dfrac{\prod_{i=1}^{n} d_i}{\prod_{i=1}^{n} w_i}$
- $d_{\text{reg}} = \sum_{i=1}^{n} (d_i - w_i) + \max\{w_j\}$
- $N_d \simeq \dfrac{1}{\prod_{i=1}^{n} w_i} \binom{n+d-1}{d}$

Overall, the complexity is divided by $\left(\prod w_i\right)^{\omega}$
when compared to a homogeneous system of the same degree.

And what about higher dimension?

For homogeneous systems in positive dimension ($m \leq n$):

- Bézout bound: $D = \prod_{i=1}^{m} d_i$
- Macaulay bound: $d_{\text{reg}} \leq \sum_{i=1}^{m}(d_i - 1) + 1$

Definition

The sequence $f_1, \ldots, f_m$ is in Noether position iff the sequence $f_1, \ldots, f_m, X_{m+1}, \ldots, X_n$ is regular.

Properties

- Information about which variables really matter to the system.
- Not necessary for homogeneous systems in "big enough" fields, because that property is always satisfied up to a linear change of variables.

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- Not necessary for homogeneous systems in “big enough” fields, because that property is always satisfied up to a linear change of variables.

Results for a positive-dimensional ideal in Noether position

Result (Faugère, Safey, V.)

Sequences in Noether pos. are generic amongst W-homo. seq. of given W-degree, assuming there exists some sequence in Noether position with that W-degree.

- Bézout bound:

$$D = \frac{\prod_{i=1}^{m} d_i}{\prod_{i=1}^{m} w_i}$$

- Macaulay bound:

$$d_{\text{reg}} = \sum_{i=1}^{m} (d_i - w_i) + \max\{w_j : j \leq m\}$$

- Algorithm matrix-F_5 still runs in complexity polynomial in the Bézout bound.
- Algorithm FGLM only works for zero-dimensional systems.
- These results are nonetheless helpful when we study affine systems (through homogenization).

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Benchmarks with generic systems

n	$\deg(I)$	t_{F_5}(qh)	Ratio for F_5	t_{FGLM}(qh)	Ratio for FGLM
7	512	0.09s	3.2	0.06s	1.7
8	1024	0.39s	4.2	0.17s	1.9
9	2048	1.63s	4.9	0.59s	2.0
10	4096	7.54s	5.4	2.36s	2.6
11	8192	33.3s	6.4	17.5s	2.4
12	16384	167.9s	6.8	115.8s	
13	32768	796.7s	8.4	782.74s	
14	65536	5040.1s	∞	5602.27s	

Benchmarks obtained with FGb on **generic** affine systems with W-degree $(4, \ldots, 4)$ for $W = (2, \ldots, 2, 1, 1)$

n	$\deg(I)$	t_{F_5}(qh)	Ratio for F_5	t_{FGLM}(qh)	Ratio for FGLM
3	16	0.00s		0.00s	
4	512	0.03s	3.7	0.07s	
5	65536	935.39s	6.9	2164.38s	3.2

Benchmarks obtained with systems arising in the DLP on Edwards curves, with W-degree (4) for $W = (2, \ldots, 2, 1)$

(Faugère, Gaudry, Huot, Renault 2013)

Conclusion

What we have done

- Theoretical results for quasi-homogeneous systems under generic hypotheses
- Variant of the usual strategy for these systems (variant of F_5 + weighted order)
- Complexity results for F_5 and FGLM for this strategy
  - Complexity overall divided by $(\prod w_i)^\omega$
  - Polynomial in the number of solutions

Perspectives

- Overdetermined systems: adapt the definitions and the results
- Affine systems: find the most appropriate system of weights

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One last word

Thanks for listening!